4.1.2 Moments and cumulants

Quite generally, the *expectation value* of any function $F(X)$ of the random variable X with outcomes $\{x_i\}$ is given by

$$\langle F(X)\rangle = \sum_i p_i F(x_i)\,. \tag{4.1.1}$$

- Of particular relevance are the *mean* or *average* of X, obtained as

  $$\langle X\rangle = \sum_i p_i x_i\,. \tag{4.1.2}$$

  Similarly, higher *moments* of the random variable are expectation values for powers of the random variable; the ℓ^{th} moment given by

  $$\langle X^\ell\rangle = \sum_i p_i x_i^\ell\,. \tag{4.1.3}$$

- A quite useful method for obtaining moments of a probability distribution function is to employ a so called *moment generating function*, which is

  $$G(\lambda) = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}\langle X^\ell\rangle = \langle e^{\lambda X}\rangle\,. \tag{4.1.4}$$

  Moments of the random variable can then be generated as terms of the coefficients of the Taylor expansion of $G(\lambda)$ around the origin.

- Another useful quantity is the *cumulant generating function* which is the logarithm of the moment characteristic function. Its expansion around the origin generates the *cumulants* of the random variable, defined through

  $$\ln G(\lambda) = \sum_{\ell=1}^{\infty} \frac{\lambda^\ell}{\ell!}\langle X^\ell\rangle_c\,. \tag{4.1.5}$$

- The first four cumulants are called the *mean*, *variance*, *skewness*, and *curtosis* (or kurtosis) of the random variable respectively, and are obtained from the moments as

  $$\begin{aligned}
  \langle X\rangle_c &= \langle X\rangle\,,\\
  \langle X^2\rangle_c &= \langle X^2\rangle - \langle X\rangle^2\,,\\
  \langle X^3\rangle_c &= \langle X^3\rangle - 3\langle X^2\rangle\langle X\rangle + 2\langle X\rangle^3\,,\\
  \langle X^4\rangle_c &= \langle X^4\rangle - 4\langle X^3\rangle\langle X\rangle - 3\langle X^2\rangle^2 + 12\langle X^2\rangle\langle X\rangle^2 - 6\langle X\rangle^4\,.
  \end{aligned} \tag{4.1.6}$$

- These relations can be inverted to give the moments in terms of the cumulants, as in

$$
\begin{aligned}
\langle X \rangle &= \langle X \rangle_c \,, \\
\left\langle X^2 \right\rangle &= \left\langle X^2 \right\rangle_c + \langle X \rangle_c^2 \,, \\
\left\langle X^3 \right\rangle &= \left\langle X^3 \right\rangle_c + 3 \left\langle X^2 \right\rangle_c \langle X \rangle_c + \langle X \rangle_c^3 \,, \\
\left\langle X^4 \right\rangle &= \left\langle X^4 \right\rangle_c + 4 \left\langle X^3 \right\rangle_c \langle X \rangle_c + 3 \left\langle X^2 \right\rangle_c^2 + 6 \left\langle X^2 \right\rangle_c \langle X \rangle_c^2 + \langle X \rangle_c^4 \,.
\end{aligned}
\tag{4.1.7}
$$

  A simple graphical representation that relates moments to cumulants will be presented later.