### 4.1.3 Bionomial distribution

Consider a random variable with two outcomes $A$ and $B$ (e.g. a coin toss) of relative probabilities $p_A$ and $p_B = 1 - p_A$. The probability that in $N$ trials the event $A$ occurs exactly $N_A$ times (e.g. 5 heads in 12 coin tosses), is given by the binomial distribution

$$p_N(N_A) = \left(\frac{N!}{N_A!(N-N_A)!}\right) p_A^{N_A} p_B^{N-N_A} \ . \tag{4.1.8}$$

The prefactor is just the coefficient obtained in the binomial expansion of $(p_A + p_B)^N$, and gives the number of possible orderings of $N_A$ events $A$ and $N_B = N - N_A$ events $B$.

The generating function for the binomial distribution is

$$G_N(\lambda) = \left\langle e^{\lambda N_A} \right\rangle = \sum_{N_A=0}^{N} \frac{N!}{N_A!(N-N_A)!} p_A^{N_A} p_B^{N-N_A} e^{\lambda N_A} = \left(p_A e^{\lambda} + p_B\right)^N \quad . \tag{4.1.9}$$

The resulting cumulant generating function is

$$\ln G_N(\lambda) = N \ln\left(p_A e^{-ik} + p_B\right) = N \ln G_1(\lambda) \, , \tag{4.1.10}$$

where $\ln G_1(\lambda)$ is the cumulant generating function for a single step. Hence, the cumulants after $N$ steps are simply $N$ times the cumulants in a single step. In each step, the allowed values of $N_A$ are 0 and 1 with respective probabilities $p_B$ and $p_A$, leading to $\left\langle N_A^\ell \right\rangle = p_A$, for all $\ell$. After $N$ trials the first two cumulants are

$$\langle N_A \rangle_c = N p_A \quad , \quad \left\langle N_A^2 \right\rangle_c = N\left(p_A - p_A^2\right) = N p_A p_B \quad . \tag{4.1.11}$$

A measure of fluctuations around the mean is provided by the *standard deviation*, which is the square root of the variance. While the mean of the binomial distribution scales as $N$, its standard deviation only grows as $\sqrt{N}$. Hence, the *relative* uncertainty becomes smaller for large $N$.

The binomial distribution is straightforwardly generalized to a *multinomial* distribution, when the several outcomes $\{A, B, \cdots, M\}$ occur with probabilities $\{p_A, p_B, \cdots, p_M\}$. The probability of finding outcomes $\{N_A, N_B, \cdots, N_M\}$ in a total of $N = N_A + N_B \cdots + N_M$ trials is

$$p_N\left(\{N_A, N_B, \cdots, N_M\}\right) = \frac{N!}{N_A! N_B! \cdots N_M!} p_A^{N_A} p_B^{N_B} \cdots p_M^{N_M} \ . \tag{4.1.12}$$