4.2.4 The Gaussian distribution

The normal (Gaussian) distribution describes a continuous real random variable x, with

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right] \quad . \tag{4.2.12}$$

The corresponding characteristic function also has a Gaussian form,

$$\tilde{p}(k) = \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-a)^2}{2\sigma^2} - ikx\right] = \exp\left[-ika - \frac{k^2\sigma^2}{2}\right] \quad . \tag{4.2.13}$$

Cumulants of the distribution can be identified from $\ln \tilde{p}(k) = -ika - k^2\sigma^2/2$, using Eq. (4.2.8), as

$$\langle x \rangle_c = a \quad , \quad \left\langle x^2 \right\rangle_c = \sigma^2 \quad , \quad \left\langle x^3 \right\rangle_c = \left\langle x^4 \right\rangle_c = \cdots = 0 \quad . \tag{4.2.14}$$

The normal distribution is thus completely specified by its two first cumulants. This makes the computation of moments using the cluster expansion of Eq, (4.2.9) particularly simple, and

$$\begin{aligned}
\langle x \rangle &= a \, , \\
\left\langle x^2 \right\rangle &= \sigma^2 + a^2 , \\
\left\langle x^3 \right\rangle &= 3\sigma^2 a + a^3 , \\
\left\langle x^4 \right\rangle &= 3\sigma^4 + 6\sigma^2 a^2 + a^4 , \quad \cdots \quad .
\end{aligned} \tag{4.2.15}$$

The normal distribution serves as the starting point for most perturbative computations in field theory. The vanishing of higher cumulants implies that all graphical computations involve only products of one point, and two point (known as propagators) clusters.