## 4.2 Continuous random variable

### 4.2.1 Probability distribution function

Let us next discuss a random variable whose allowed values are not discrete, but continuous. In particular, consider a random variable $x$, whose outcomes are real numbers, i.e. $\mathcal{S} \in \{-\infty < x < \infty\}$.

- *The cumulative probability function* (CPF) $P(x)$, is the probability of an outcome with *any value* less than $x$, i.e. $P(x) = \text{prob.}(E \subset [-\infty, x])$. $P(x)$ must be a monotonically increasing function of $x$, with $P(-\infty) = 0$ and $P(+\infty) = 1$.

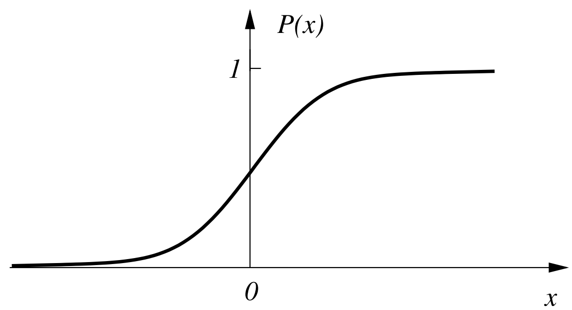


Figure 4.1: A typical cumulative probability function.

- *The probability density function* (PDF) is defined by $p(x) \equiv dP(x)/dx$. Hence, $p(x)dx = \text{prob.}(E \in [x, x + dx])$. As a probability density, it is *positive*, and normalized such that

$$\text{prob.}(\mathcal{S}) = \int_{-\infty}^{\infty} dx \; p(x) = 1 \; . \tag{4.2.1}$$

Note that since $p(x)$ is a *probability density*, it has dimensions of $[x]^{-1}$, and changes its value if the units measuring $x$ are modified. Unlike $P(x)$, the PDF has no upper bound, i.e. $0 < p(x) < \infty$, and may contain divergences as long as they are integrable.

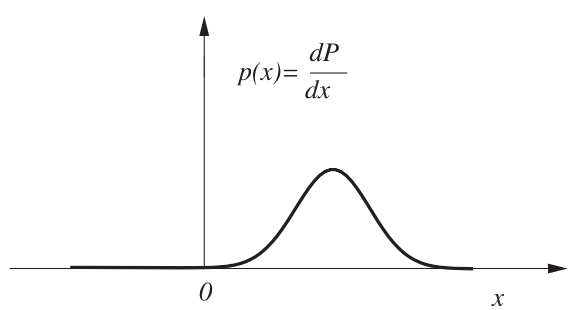


Figure 4.2: A typical probability density function.

### 4.2.2 Change of variables

- Consider a function $F(X)$ of the random variable $X$. as before the *expectation value* of the function is given by

$$\langle F(x) \rangle = \int_{-\infty}^{\infty} dx \; p(x) F(x) \; . \tag{4.2.2}$$

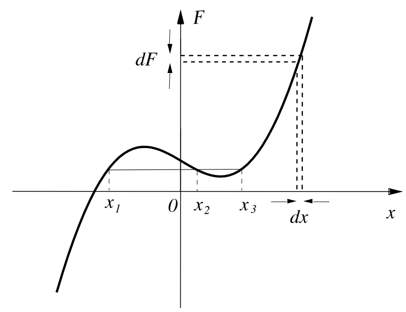


Figure 4.3: Obtaining the PDF for the function $F(x)$.

However, the function $F$ is itself a random variable, with an associated PDF of $p_F(f)df = \text{prob.}(F \in [f, f+df])$. There may be multiple solutions $x_i$, to the equation $F(x) = f$, and

$$p_F(f)df = \sum_i p(x_i) dx_i \,, \quad \Longrightarrow \quad p_F(f) = \sum_i p(x_i) \left| \frac{dx}{dF} \right|_{x=x_i} \quad . \tag{4.2.3}$$

The factors of $|dx/dF|$ are the *Jacobians* associated with the change of variables from $x$ to $F$.

- As an example, consider $p(x) = \lambda \exp(-\lambda|x|)/2$, and the function $F(x) = x^2$. There are two solutions to $F(x) = f$, located at $x_\pm = \pm\sqrt{f}$, with corresponding Jacobians $|\pm f^{-1/2}/2|$. Hence,

$$P_F(f) = \frac{\lambda}{2} \exp\left(-\lambda\sqrt{f}\right) \left( \left| \frac{1}{2\sqrt{f}} \right| + \left| \frac{-1}{2\sqrt{f}} \right| \right) = \frac{\lambda \exp\left(-\lambda\sqrt{f}\right)}{2\sqrt{f}} \,,$$

for $f > 0$, and $p_F(f) = 0$ for $f < 0$. Note that $p_F(f)$ has an (integrable) divergence at $f = 0$.

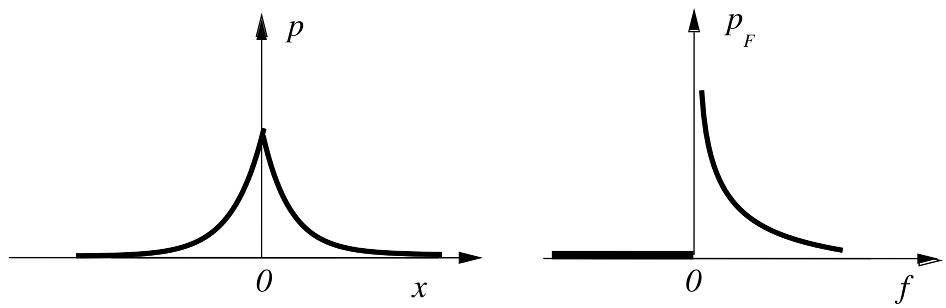


Figure 4.4: Probability density functions for $x$, and $F = x^2$.

### 4.2.3 The characteristic function

As in the earlier case of a discrete random variable, we can define the moment generating function for a continuous variable as $\langle G(\lambda) = \langle e^{\lambda X} \rangle$. The *characteristic function* is simply the moment generating function with $\lambda = -ik$, i.e.

$$\tilde{p}(k) = \left\langle e^{-ikx} \right\rangle = \int dx \; p(x) \; e^{-ikx} \,. \tag{4.2.4}$$

The notation $\tilde{p}(k)$ is used to emphasize that the characteristic function is in fact the Fourier transform of the original PDF. The PDF can thus be recovered from the characteristic function through the inverse Fourier transform, i.e.

$$p(x) = \frac{1}{2\pi} \int dk \tilde{p}(k) \; e^{+ikx} \,. \tag{4.2.5}$$

*Moments* of the random variable are obtained by expanding $\tilde{p}(k)$ in powers of $k$ around $k = 0$, as

$$\tilde{p}(k) = \left\langle \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} x^n \right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle \quad . \tag{4.2.6}$$

Moments of the PDF around any point $x_0$ can also be generated using the expansion

$$e^{ikx_0} \tilde{p}(k) = \left\langle e^{-ik(x-x_0)} \right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle (x - x_0)^n \rangle \quad . \tag{4.2.7}$$

*Cumulants* of the random variable are obtained from a corresponding expansion of $\ln \tilde{p}(k)$, as

$$\ln \tilde{p}(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c \quad . \tag{4.2.8}$$

An important theorem allows easy computation of moments in terms of the cumulants: Represent the $n^{\text{th}}$ cumulant graphically as a *connected cluster* of $n$ points. The $m^{\text{th}}$ moment is then obtained by summing all possible subdivisions of $m$ points into groupings of smaller (connected or disconnected) clusters. The contribution of each subdivision to the sum is the product of the connected cumulants that it represents. Using this result the first four moments are computed graphically as

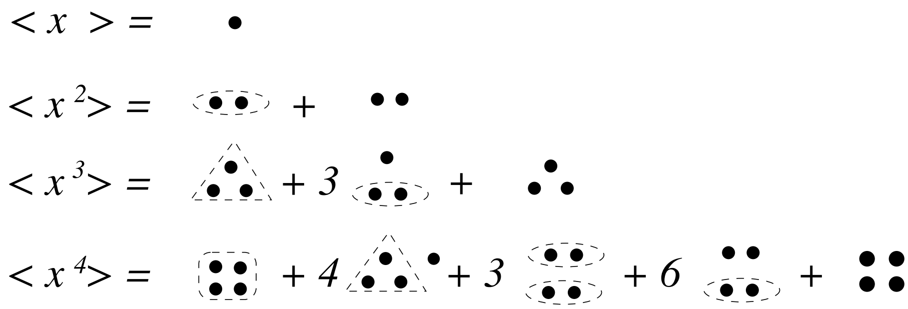


Figure 4.5: Graphical computation of the first four moments.

The corresponding algebraic expressions are

$$\begin{aligned}
\langle x\rangle &= \langle x\rangle_c\,, \\
\left\langle x^2\right\rangle &= \left\langle x^2\right\rangle_c + \langle x\rangle_c^2\,, \\
\left\langle x^3\right\rangle &= \left\langle x^3\right\rangle_c + 3\left\langle x^2\right\rangle_c\langle x\rangle_c + \langle x\rangle_c^3\,, \\
\left\langle x^4\right\rangle &= \left\langle x^4\right\rangle_c + 4\left\langle x^3\right\rangle_c\langle x\rangle_c + 3\left\langle x^2\right\rangle_c^2 + 6\left\langle x^2\right\rangle_c\langle x\rangle_c^2 + \langle x\rangle_c^4\,.
\end{aligned} \tag{4.2.9}$$

This theorem, which is the starting point for various diagrammatic computations in statistical mechanics and field theory, is easily proved by equating the expressions in Eqs. (4.2.6) and (4.2.8) for $\tilde{p}(k)$

$$\sum_{m=0}^{\infty}\frac{(-ik)^m}{m!}\langle x^m\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{(-ik)^n}{n!}\langle x^n\rangle_c\right] = \prod_n\sum_{p_n}\left[\frac{(-ik)^{np_n}}{p_n!}\left(\frac{\langle x^n\rangle_c}{n!}\right)^{p_n}\right]. \tag{4.2.10}$$

Matching the powers of $(-ik)^m$ on the two sides of the above expression leads to

$$\langle x^m\rangle = \sum_{\{p_n\}}{}' \, m!\prod_n\frac{1}{p_n!(n!)^{p_n}}\langle x^n\rangle_c^{p_n}\ . \tag{4.2.11}$$

The sum is restricted such that $\sum np_n = m$, and leads to the graphical interpretation given above, as the numerical factor is simply the number of ways of breaking $m$ points into $\{p_n\}$ clusters of $n$ points.

### 4.2.4 The Gaussian distribution

*The normal (Gaussian) distribution* describes a continuous real random variable $x$, with

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-a)^2}{2\sigma^2}\right]\quad . \tag{4.2.12}$$

The corresponding characteristic function also has a Gaussian form,

$$\tilde{p}(k) = \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-a)^2}{2\sigma^2} - ikx\right] = \exp\left[-ika - \frac{k^2\sigma^2}{2}\right] \quad . \tag{4.2.13}$$

Cumulants of the distribution can be identified from $\ln \tilde{p}(k) = -ika - k^2\sigma^2/2$, using Eq. (4.2.8), as

$$\langle x \rangle_c = a \quad , \quad \left\langle x^2 \right\rangle_c = \sigma^2 \quad , \quad \left\langle x^3 \right\rangle_c = \left\langle x^4 \right\rangle_c = \cdots = 0 \quad . \tag{4.2.14}$$

The normal distribution is thus completely specified by its two first cumulants. This makes the computation of moments using the cluster expansion of Eq, (4.2.9) particularly simple, and

$$\begin{aligned}
\langle x \rangle &= a\,, \\
\left\langle x^2 \right\rangle &= \sigma^2 + a^2, \\
\left\langle x^3 \right\rangle &= 3\sigma^2 a + a^3, \\
\left\langle x^4 \right\rangle &= 3\sigma^4 + 6\sigma^2 a^2 + a^4, \quad \cdots \quad .
\end{aligned} \tag{4.2.15}$$

The normal distribution serves as the starting point for most perturbative computations in field theory. The vanishing of higher cumulants implies that all graphical computations involve only products of one point, and two point (known as propagators) clusters.