4.3 Many random variables

4.3.1 Joint PDF

With more than one random variable, the set of outcomes is an N-dimensional space, $\mathcal{S}_{\mathbf{x}} = \{-\infty < x_1, x_2, \cdots, x_N < \infty\}$. For example, describing the location and velocity of a gas particle requires six coordinates.

- *The joint PDF* $p(\mathbf{x})$, is the probability density of an outcome in a volume element $d^N\mathbf{x} = \prod_{i=1}^N dx_i$ around the point $\mathbf{x} = \{x_1, x_2, \cdots, x_N\}$. The joint PDF is normalized such that

$$p_{\mathbf{x}}(\mathcal{S}) = \int d^N\mathbf{x}\, p(\mathbf{x}) = 1\ . \tag{4.3.1}$$

  If, and only if, the N random variables are *independent*, the joint PDF is the product of individual PDFs,

$$p(\mathbf{x}) = \prod_{i=1}^{N} p_i(x_i) \quad . \tag{4.3.2}$$

- *The unconditional PDF* describes the behavior of a subset of random variables, independent of the values of the others. For example, if we are interested only in the location of a gas particle, an unconditional PDF can be constructed by integrating over all velocities at a given location, $p(\vec{x}) = \int d^3\vec{v}\, p(\vec{x}, \vec{v})$; more generally

$$p(x_1, \cdots, x_m) = \int \prod_{i=m+1}^{N} dx_i\, p(x_1, \cdots, x_N)\ . \tag{4.3.3}$$

- *The conditional PDF* describes the behavior of a subset of random variables, for specified values of the others. For example, the PDF for the velocity of a particle at a particular location $\vec{x}$, denoted by $p(\vec{v} \mid \vec{x})$, is proportional to the joint PDF $p(\vec{v} \mid \vec{x}) = p(\vec{x}, \vec{v})/\mathcal{N}$. The constant of proportionality, obtained by normalizing $p(\vec{v} \mid \vec{x})$, is

$$\mathcal{N} = \int d^3\vec{v}\, p(\vec{x}, \vec{v}) = p(\vec{x})\,, \tag{4.3.4}$$

  the unconditional PDF for a particle at $\vec{x}$. In general, the unconditional PDFs are obtained from *Bayes' theorem* as

$$p(x_1, \cdots, x_m \mid x_{m+1}, \cdots, x_N) = \frac{p(x_1, \cdots, x_N)}{p(x_{m+1}, \cdots, x_N)} \quad . \tag{4.3.5}$$

  Note that if the random variables are independent, the unconditional PDF is equal to the conditional PDF.