4.3.3 Multi-variable Gaussian

The generalization of Eq. (4.2.12) to N random variables takes the form

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \det[C]}} \exp\left[-\frac{1}{2}\sum_{mn}\left(C^{-1}\right)_{mn}(x_m - a_m)(x_n - a_n)\right] \quad , \tag{4.3.10}$$

where C is a symmetric matrix, and C^{-1} is its inverse. The simplest way to get the normalization factor is to make a linear transformation from the variables $y_j = x_j - a_j$, using the unitary matrix that diagonalizes C. This reduces the normalization to that of the product of N Gaussians whose variances are determined by the eigenvalues of C. The product of the eigenvalues is the determinant $\det[C]$. (This also indicates that the matrix C must be positive definite.)

The corresponding joint characteristic function is obtained by similar manipulations, and is given by

$$\tilde{p}(\mathbf{k}) = \exp\left[-ik_m a_m - \frac{1}{2}C_{mn}k_m k_n\right] \quad , \tag{4.3.11}$$

where the summation convention (implicit summation over a repeated index) is used.

The joint cumulants of the Gaussian are then obtained from $\ln\tilde{p}(\mathbf{k})$ as

$$\langle x_m\rangle_c = a_m \quad , \quad \langle x_m * x_n\rangle_c = C_{mn} \quad , \tag{4.3.12}$$

with all higher cumulants equal to zero. In the special case of $\{a_m\} = 0$, all odd *moments* of the distribution are zero, while the general rules for relating moments to cumulants indicate that any even moment is obtained by summing over all ways of grouping the involved random variables into pairs, e.g.

$$\langle x_a x_b x_c x_d\rangle = C_{ab}C_{cd} + C_{ac}C_{bd} + C_{ad}C_{bc}. \tag{4.3.13}$$

In the context of field theories, this result is referred to as *Wick's theorem*.