4.3 Many random variables

4.3.1 Joint PDF

With more than one random variable, the set of outcomes is an N-dimensional space, $\mathcal{S}_{\mathbf{x}} = \{-\infty < x_1, x_2, \cdots, x_N < \infty\}$. For example, describing the location and velocity of a gas particle requires six coordinates.

- *The joint PDF* $p(\mathbf{x})$, is the probability density of an outcome in a volume element $d^N\mathbf{x} = \prod_{i=1}^N dx_i$ around the point $\mathbf{x} = \{x_1, x_2, \cdots, x_N\}$. The joint PDF is normalized such that
$$p_{\mathbf{x}}(\mathcal{S}) = \int d^N\mathbf{x}\ p(\mathbf{x}) = 1\ . \tag{4.3.1}$$
If, and only if, the N random variables are *independent*, the joint PDF is the product of individual PDFs,
$$p(\mathbf{x}) = \prod_{i=1}^N p_i(x_i) \quad . \tag{4.3.2}$$

- *The unconditional PDF* describes the behavior of a subset of random variables, independent of the values of the others. For example, if we are interested only in the location of a gas particle, an unconditional PDF can be constructed by integrating over all velocities at a given location, $p(\vec{x}) = \int d^3\vec{v}\ p(\vec{x}, \vec{v}\,)$; more generally
$$p(x_1, \cdots, x_m) = \int \prod_{i=m+1}^N dx_i\ p(x_1, \cdots, x_N)\ . \tag{4.3.3}$$

- *The conditional PDF* describes the behavior of a subset of random variables, for specified values of the others. For example, the PDF for the velocity of a particle at a particular location $\vec{x}$, denoted by $p(\vec{v} \mid \vec{x}\,)$, is proportional to the joint PDF $p(\vec{v} \mid \vec{x}\,) = p(\vec{x}, \vec{v}\,)/\mathcal{N}$. The constant of proportionality, obtained by normalizing $p(\vec{v} \mid \vec{x}\,)$, is
$$\mathcal{N} = \int d^3\vec{v}\ p(\vec{x}, \vec{v}) = p(\vec{x}\,)\,, \tag{4.3.4}$$
the unconditional PDF for a particle at $\vec{x}$. In general, the unconditional PDFs are obtained from *Bayes' theorem* as
$$p(x_1, \cdots, x_m \mid x_{m+1}, \cdots, x_N) = \frac{p(x_1, \cdots, x_N)}{p(x_{m+1}, \cdots, x_N)} \quad . \tag{4.3.5}$$
Note that if the random variables are independent, the unconditional PDF is equal to the conditional PDF.

4.3.2 Joint moments and cumulants

- *The expectation value* of a function $F(\mathbf{x})$, is obtained as before from

$$\langle F(\mathbf{x})\rangle = \int d^N\mathbf{x}\; p(\mathbf{x})F(\mathbf{x})\ . \tag{4.3.6}$$

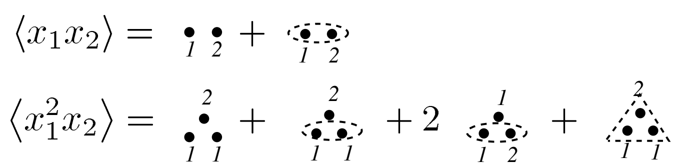
- *The joint characteristic function* is obtained from the N-dimensional Fourier transformation of the joint PDF,

$$\tilde{p}(\mathbf{k}) = \left\langle \exp\left(-i\sum_{j=1}^{N} k_j x_j\right)\right\rangle \quad . \tag{4.3.7}$$

- The *joint moments* and *joint cumulants* are generated by $\tilde{p}(\mathbf{k})$ and $\ln\tilde{p}(\mathbf{k})$ respectively, as

$$\begin{aligned}
\langle x_1^{n_1} x_2^{n_2}\cdots x_N^{n_N}\rangle &= \left[\frac{\partial}{\partial(-ik_1)}\right]^{n_1}\left[\frac{\partial}{\partial(-ik_2)}\right]^{n_2}\cdots\left[\frac{\partial}{\partial(-ik_N)}\right]^{n_N}\tilde{p}(\mathbf{k}=\mathbf{0}) \quad ,\\
\langle x_1^{n_1} * x_2^{n_2} *\cdots x_N^{n_N}\rangle_c &= \left[\frac{\partial}{\partial(-ik_1)}\right]^{n_1}\left[\frac{\partial}{\partial(-ik_2)}\right]^{n_2}\cdots\left[\frac{\partial}{\partial(-ik_N)}\right]^{n_N}\ln\tilde{p}(\mathbf{k}=\mathbf{0}) \quad .
\end{aligned} \tag{4.3.8}$$

- The previously described graphical relation between joint moments (all clusters of labeled points), and joint cumulant (connected clusters) is still applicable. For example, from



we obtain

$$\begin{aligned}
\langle x_1x_2\rangle &= \langle x_1\rangle_c\langle x_2\rangle_c + \langle x_1 * x_2\rangle_c \quad , \quad \text{and}\\
\left\langle x_1^2x_2\right\rangle &= \langle x_1\rangle_c^2\langle x_2\rangle_c + \left\langle x_1^2\right\rangle_c\langle x_2\rangle_c + 2\langle x_1 * x_2\rangle_c\langle x_1\rangle_c + \left\langle x_1^2 * x_2\right\rangle_c \quad .
\end{aligned} \tag{4.3.9}$$

The *connected correlation* $\langle x_\alpha * x_\beta\rangle_c$, is zero if x_α and x_β are independent random variables.

4.3.3 Multi-variable Gaussian

The generalization of Eq. (4.2.12) to N random variables takes the form

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \det[C]}} \exp\left[-\frac{1}{2}\sum_{mn}\left(C^{-1}\right)_{mn}(x_m - a_m)(x_n - a_n)\right] \quad , \tag{4.3.10}$$

where C is a symmetric matrix, and C^{-1} is its inverse. The simplest way to get the normalization factor is to make a linear transformation from the variables $y_j = x_j - a_j$, using the unitary matrix that diagonalizes C. This reduces the normalization to that of the product of N Gaussians whose variances are determined by the eigenvalues of C. The product of the eigenvalues is the determinant $\det[C]$. (This also indicates that the matrix C must be positive definite.)

The corresponding joint characteristic function is obtained by similar manipulations, and is given by

$$\tilde{p}(\mathbf{k}) = \exp\left[-ik_m a_m - \frac{1}{2}C_{mn}k_m k_n\right] \quad , \tag{4.3.11}$$

where the summation convention (implicit summation over a repeated index) is used.

The joint cumulants of the Gaussian are then obtained from $\ln\tilde{p}(\mathbf{k})$ as

$$\langle x_m \rangle_c = a_m \quad , \quad \langle x_m * x_n \rangle_c = C_{mn} \quad , \tag{4.3.12}$$

with all higher cumulants equal to zero. In the special case of $\{a_m\} = 0$, all odd *moments* of the distribution are zero, while the general rules for relating moments to cumulants indicate that any even moment is obtained by summing over all ways of grouping the involved random variables into pairs, e.g.

$$\langle x_a x_b x_c x_d \rangle = C_{ab}C_{cd} + C_{ac}C_{bd} + C_{ad}C_{bc}. \tag{4.3.13}$$

In the context of field theories, this result is referred to as *Wick's theorem*.