### 4.4.3 Stirling's approximation

A highly useful approximation for $N!$ at large $N$ can be obtained by using a variant of the above method of integration. In order to get an integral representation of $N!$, start with the result

$$\int_0^\infty dx e^{-\alpha x} = \frac{1}{\alpha}\,. \tag{4.4.15}$$

Repeated differentiation of both sides of the above equation with respect to $\alpha$ leads to

$$\int_0^\infty dx\, x^N e^{-\alpha x} = \frac{N!}{\alpha^{N+1}}\,. \tag{4.4.16}$$

Although the above result only applies to integer $N$, it is possible to define by analytical continuation a function,

$$\Gamma(N+1) \equiv N! = \int_0^\infty dx x^N e^{-x}\,, \tag{4.4.17}$$

for all $N$. While the integral in Eq. (4.4.17) is not exactly in the form of Eq. (4.4.11), it can still be evaluated by a similar method. The integrand can be written as $\exp\big(N\phi(x)\big)$, with $\phi(x) = \ln x - x/N$. The exponent has a maximum at $x_{\rm max} = N$, with $\phi(x_{\rm max}) = \ln N - 1$, and $\phi''(x_{\rm max}) = -1/N^2$. Expanding the integrand in Eq. (4.4.17) around this point yields,

$$N! \approx \int dx \exp\left[N\ln N - N - \frac{1}{2N}(x-N)^2\right] \approx N^N e^{-N}\sqrt{2\pi N}\,, \tag{4.4.18}$$

where the integral is evaluated by extending its limits to $[-\infty, \infty]$. Stirling's formula is obtained by taking the logarithm of Eq. (4.4.18) as,

$$\ln N! = N\ln N - N + \frac{1}{2}\ln(2\pi N) + \mathcal{O}\left(\frac{1}{N}\right)\,. \tag{4.4.19}$$