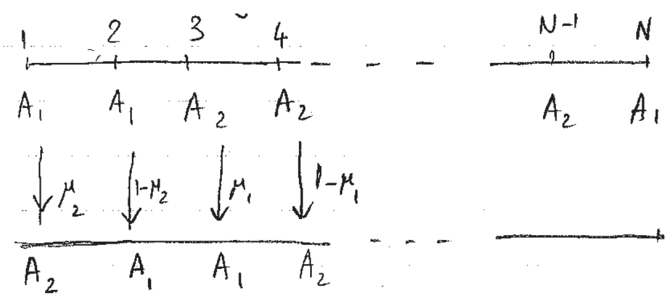
5.1.3 Evolving binary sequence



As a simple example, consider a *binary* sequence (i.e. $m = 2$) with independent states A_1 or A_2 at each site.[2] Let us assume that the state A_1 can "mutate" to A_2 at a rate μ_2, while state A_2 may change to A_1 with a rate μ_1. The probabilities $p_1(t)$ and $p_2(t)$ now evolve in time as

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 & -\mu_1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{5.1.7}$$

The above 2×2 transition rate matrix has the following two eigenvectors

$$\begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 & -\mu_1 \end{pmatrix}\begin{pmatrix} \frac{\mu_1}{\mu_1+\mu_2} \\ \frac{\mu_2}{\mu_1+\mu_2} \end{pmatrix} = 0, \quad \text{and} \quad \begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 & -\mu_1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -(\mu_1+\mu_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.1.8}$$

As anticipated, there is an eigenvector $\overrightarrow{p^*}$ with eigenvalue of zero; the elements of this vector have been normalized to add to unity, as required for probabilities. We have not normalized the second eigenvector, whose eigenvalue $-(\mu_1 + \mu_2)$ determines the rate of approach to steady state.

To demonstrate evolution of probabilities with time, let us start with a sequence that is purely A_1, i.e. with $p_1 = 1$ and $p_2 = 0$ at $t = 0$. The formal solution to the linear differential equation (5.1.7) is

$$\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \exp\left[t\begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 & -\mu_1 \end{pmatrix}\right]\begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix}. \tag{5.1.9}$$

Decomposing the initial state as a sum over the eigenvectors, and noting the action of the rate matrix on each eigenvector from Eq. (5.1.8), we find

$$\begin{aligned}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \exp\left[t\begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 & -\mu_1 \end{pmatrix}\right]\left[\begin{pmatrix} \frac{\mu_1}{\mu_1+\mu_2} \\ \frac{\mu_2}{\mu_1+\mu_2} \end{pmatrix} + \frac{\mu_2}{\mu_1+\mu_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \\ &= \begin{pmatrix} \frac{\mu_1}{\mu_1+\mu_2} + e^{-(\mu_1+\mu_2)t}\frac{\mu_2}{\mu_1+\mu_2} \\ \frac{\mu_2}{\mu_1+\mu_2} - e^{-(\mu_1+\mu_2)t}\frac{\mu_2}{\mu_1+\mu_2} \end{pmatrix}.\end{aligned} \tag{5.1.10}$$

[2] Clearly with the assumption of independence we are really treating independent sites, and the insistence on a sequence may appear frivolous. The advantage of this perspective, however, will become apparent in the next section.

At long times the probabilities to find state A_1 or A_2 are in the ratios μ_1 to μ_2 as dictated by the steady state eigenvector. The rate at which the probabilities converge to this steady steady is determined by the second eigenvalue $-(\mu_1 + \mu_2)$.