5.1.4 The Master equation

In many circumstances of interest the probabilities change slowly and continuously over time, in which case we introduce a time interval Δt between subsequent generations, and write

$$\frac{p_\alpha(\tau+1) - p_\alpha(\tau)}{\Delta t} = \sum_{\beta \neq \alpha} \left[\frac{\pi_{\alpha\beta}}{\Delta t} p_\beta(\tau) - \frac{\pi_{\beta\alpha}}{\Delta t} p_\alpha(\tau) \right]. \tag{5.1.11}$$

In the limit of small Δt, $[p_\alpha(\tau+1) - p_\alpha(\tau)]/\Delta t \approx dp_\alpha/dt$, while

$$\frac{\pi_{\alpha\beta}}{\Delta t} = R_{\alpha\beta} + \mathcal{O}(\Delta t) \quad \text{for } \alpha \neq \beta, \tag{5.1.12}$$

are the off-diagonal elements of the matrix $\mathbf{R}$ of *transition probability rates*. The diagonal elements of the matrix describe the depletion rate of a particular state, and by conservation of probability must satisfy, as in Eq. (5.1.2),

$$\sum_\alpha R_{\alpha\beta} = 0, \quad \text{or} \quad R_{\beta\beta} = -\sum_{\alpha \neq \beta} R_{\alpha\beta}. \tag{5.1.13}$$

We thus arrive at

$$\frac{dp_\alpha(t)}{dt} = \sum_{\beta \neq \alpha} \left(R_{\alpha\beta} p_\beta(t) - R_{\beta\alpha} p_\alpha(t) \right) \quad , \tag{5.1.14}$$

which is known as the *Master equation*.