### 5.1.5 Mutating Population

The previous example of a binary sequence of length $N$ can be recast and interpreted in terms of the evolution of a (haploid) population of fixed size as follows. Let us assume that $A_1$ and $A_2$ denote two forms of a particular allele. In each generation any individual is replaced by an offspring that mostly retains its progenitor's allele, but may mutate to the other form at some rate. In this model the total population size is fixed to $N$, while the sub-populations $N_1$ and $N_2$ may vary. A particular state of the population is thus described by $N_1 = n$ and $N_2 = N - n$, and since $n = 0, 1, \cdots, N$ there are $N+1$ possible states. At a particular time, the system may be in any one of these states with probability $p(n,t)$, and we would like to follow the evolution of these probabilities.

After an individual replication event ($A_1$ to $A_1$ at rate $-\mu_2$, $A_1$ to $A_2$ at rate $\mu_2$, $A_2$ to $A_1$ at rate $\mu_1$, or $A_2$ to $A_2$ at rate $-\mu_1$), the number $N$ either stays the same, or changes by unity. Thus the transition rate matrix only has non-zero terms along or adjoining to the diagonal. For example

$$R_{n,n+1} = \mu_2(n+1), \quad \text{and} \quad R_{n,n-1} = \mu_1(N - n + 1), \qquad (5.1.15)$$

where the former indicates that a population of $n + 1$ $A_1$s can decrease by unity if any one of them mutates to $A_2$, while a population with $n - 1$ $A_1$s increases by unity if any of $A_2$s mutates to $A_1$. The diagonal terms are obtained from the normalization condition in Eq. (5.1.13) resulting in the Master equation

$$\frac{dp(n,t)}{dt} = \mu_2(n+1)p(n+1) + \mu_1(N-n+1)p(n-1) - \mu_2 n p(n) - \mu_1(N-n)p(n), \quad (5.1.16)$$

for $0 < n < N$, and with boundary terms

$$\frac{dp(0,t)}{dt} = \mu_2 p(1) - \mu_1 N p(0), \quad \text{and} \quad \frac{dp(N,t)}{dt} = \mu_1 p(N-1) - \mu_2 N p(N). \qquad (5.1.17)$$

The above equation looks complicated, but in fact admits a simple solution as justified in the different context described next.