## 1.2 First order ordinary differential equations

### 1.2.1 General solution

Equation (1.1.10) is a typical first order ordinary differential equation (ODE). The stationary points of a general such ODE of the form $\dot{x} = F(x)$ are obtained by as solutions to $F(x^*) = 0$. To plot the trajectory of motion, starting from $x(t = 0) = x_0$, we can use the following procedure[2]

$$\frac{dx}{dt} = F(x), \quad \Longrightarrow \quad \frac{dx}{F(x)} = dt, \quad \Longrightarrow \quad t = \int_{x_0}^{x(t)} \frac{dx'}{F(x')} . \tag{1.2.3}$$

Assuming that we can evaluate the integral in the final expression, we still need to invert the result to obtain the explicit form of $x(t)$.

Let us redrive the solution for the decaying linear spring in Eq. (1.1.11), with $F(x) = -\gamma x$, noting:

$$t = \int_{x_0}^{x(t)} \frac{dx'}{(-\gamma x')} = -\frac{1}{\gamma} \ln\left(\frac{x(t)}{x_0}\right), \quad \Longrightarrow \quad x(t) = x_0 e^{-\gamma t} , \tag{1.2.4}$$

as obtained in Eq. (1.1.15) by summing the Taylor series.

A slight variations is obtained by considering $F(x) = u - \gamma x$, starting from $x = 0$ and $t = 0$, in which case

$$t = \int_{0}^{x(t)} \frac{dx'}{(u - \gamma x')} = -\frac{1}{\gamma} \ln\left(\frac{u - \gamma x(t)}{u}\right), \quad \Longrightarrow \quad x(t) = \frac{u}{\gamma}\left(1 - e^{-\gamma t}\right) . \tag{1.2.5}$$

Once again, the coordinate approaches its equilibrium point, $x^* = u/\gamma$ (solution of $F(x^*) = 0$ exponentially. Indeed this solution is identical to the previous one, obtained by a shifting the variable $x$ by $x^* = x_0$.

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[2]To justify the first step in Eq. (1.2.3), recall that from the Taylor series

$$x(t + dt) = x(t) + \frac{dx}{dt} dt + \text{terms of order } dt^2 \text{ and higher.} \tag{1.2.1}$$

Thus to lowest order in $dt$ as $dt \to 0$,

$$dx \equiv x(t + dt) - x(t) = dt\, \frac{dx}{dt} \quad \Longrightarrow \quad dt = \frac{dx}{dx/dt} . \tag{1.2.2}$$