1.2.2 Population growth – Transcritical bifurcation

After an organism invades a hospitable habitat, its initial rapid reproduction is typically exponential. However, competition for available resources eventually slows down growth. Indicating the size of the population by $N(t)$, and tis rate of change by $G(N)$ leads to the differential equation

$$\frac{dN}{dt} = G(N) = g_0 + g_1 N + g_2 N^2 + \cdots . \tag{1.2.6}$$

The terms in the Taylor expansion of $G(N)$ can now be constrained as applicable to description of a population:

- $g_0 = 0$ if the only mechanism for population growth is reproduction of its existing members.
- $g_1 \equiv r > 0$ is the reproduction rate that by itself leads to exponential growth. In principle, this parameter can be made negative, e.g. by removing needed resources.
- $g_2 < 0$, since competition for resources reduces the reproduction rate. With a view to later interpretation we relabel $g_2 = -r/N^*$.

Typically, the Taylor expansion of $G(N)$ is stopped at second order, resulting in

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N^*}\right) , \tag{1.2.7}$$

with N^* denoting the capacity of the habitat. Following the general procedure, the solution to this ODE can be obtained as follows

$$t = \int_{N_0}^{N(t)} \frac{dN'}{rN'(1 - N'/N^*)} = \frac{1}{r}\int_{N_0}^{N(t)} dN' \left[\frac{1}{N'} + \frac{1}{N^* - N'}\right] . \tag{1.2.8}$$

Note that we can always turn a composite fraction into a sum of fractions involving its factors. this enables easy computation of the integrals, leading to

$$t = \frac{1}{r}\ln\frac{N(t)(N^* - N_0)}{N_0(N^* - N(t))} \quad \Longrightarrow \quad \frac{N(t)}{(N^* - N(t))} = \frac{N_0}{(N^* - N_0)}e^{rt} . \tag{1.2.9}$$

An easy way to solve the final equation is by noting that any fraction $A/B = C/D$ can be recast as $A/(A + B) = C/(C + D)$. Applying this to the above equation leads to

$$\frac{N(t)}{N^*} = \frac{N_0 e^{rt}}{N^* + N_0(e^{rt} - 1)} \quad \Longrightarrow \quad N(t) = \frac{N^* N_0 e^{rt}}{N^* + N_0(e^{rt} - 1)} . \tag{1.2.10}$$

In the context of population evolution, Eq. (1.2.7) is known as the *logistic growth*; its solution in Eq. (1.2.10) is the *logistic growth law.*

Now consider initial populations of the same size N_0 introduced in a range of habitats differing in the parameter r in Eq. (1.2.7); hospitable habitats correspond to $r > 0$, inhospitable ones to $r < 0$. In considering both positive and negative r it is more natural to write the growth equation as $\dot{N} = rN - g_2 N^2$. For the same value of g_2 (degree of competition), as $r \to 0$, the steady state (long-time) size of the population also vanishes as $N^* = r/g_2$. Clearly, for $r < 0$, a negative solution for N^* is meaningless, and the population collapses. Upon changing the variable r, the function $N^*(r)$ thus changes from $N^* = 0$ for $r < 0$ to $N^* = r/g_2$ for $r > 0$. Such a function that cannot be represented by a single Taylor series around the point $r = 0$, is said to be *non-analytic* at that point. This particular non-analyticity of $N^*(r)$ is an example of a so-called *bifurcation*. It is best understood by noting that Eq. (1.2.7) represents gradient descent in the "potential" $V(N) = -rN^2/2 + g_2 N^3/3$. The potential has two extrema at $N^* = 0$ and $N^* = r/g_2$. For $r < 0$, the solution at $N^* = 0$ is stable, while for $r > 0$, the one at $N^* = r/g_2$; at $r = 0$, the two extrema collide and change stability.[3]

The above scenario, generic to ODEs of the form

$$\dot{y} = \epsilon y - y^2 \,. \qquad (1.2.11)$$

is known as a *transcritical* bifurcation. Note that the Logistic Eq. (1.2.7) is equivalent to Eq.(1.2.11) upon rescaling $N = N^* y/r$ and setting $\epsilon = r$.

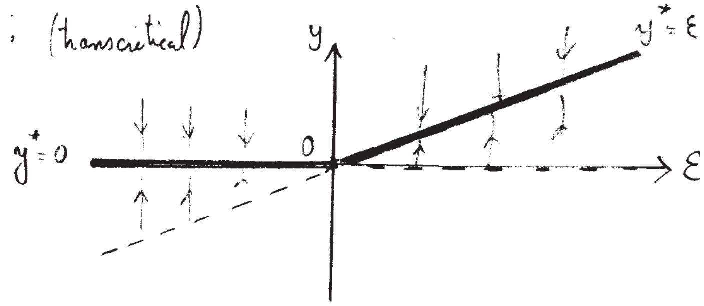

Figure 1.1: Graphical representation of Eq. (1.2.11) with arrows pointing to the direction of $\dot{y}$.

[3]The solutions to $dV/dx|_{x^*} = 0$ correspond to extrema of the function $V(x)$, including both minima and maxima. In a dynamics represented by gradient descent in $V(x)$, the minima represent *stable* end-points while the maxima are *unstable* points. Any small perturbation around a maximum will cause the coordinate to descent to the closest minimum.