1.2.3 Symmetry breaking – Pitchfork bifurcation

In a *generic* Taylor expansion, all terms in the series must be present, unless excluded for a specific reason. For example, the zeroth order term in Eq. (1.2.6) is not present in Eq. (1.2.7) given the requirement that population growth comes only from reproduction of existing members. The linear term in this equation should generically be present, although its sign can in principle be modified by changing external conditions. The bifurcation point at which $g_1 = 0$ is a prominent example of a special non-generic point in the space of model parameters.

Similarly, the quadratic term in Eq. (1.2.11) is generically present, unless forbidden by a fundamental requirement. In many cases, symmetry considerations constrain mathematical descriptions. In particular, for a system with the reflection $y \to -y$, only odd powers of y can appear in the equation for $\dot{y}$, in which case the low order Taylor expansion in Eq. (1.2.11) has to be replaced with

$$\dot{y} = \epsilon y - y^3 \,. \tag{1.2.12}$$

This equation describes gradient descent in a potential $V(y) = -\epsilon y^2/2 + y^4/3$. For $\epsilon < 0$ the only minimum (stationary) point is at $y^* = 0$, while for $\epsilon > 0$, $y^* = 0$ becomes a maximum (unstable) and a pair of stable fixed points appear at $\pm\sqrt{\epsilon}$. The transition at $\epsilon = 0$ is a *(supercritical) pitchfork bifurcation*, and the choice of one or the other stable point is by *spontaneous symmetry breaking*. Symmetry breaking is a very important concept in physics, occurring in all realms from elementary particles to condensed matter physics (for example ferromagnetism) to cosmology.[4]

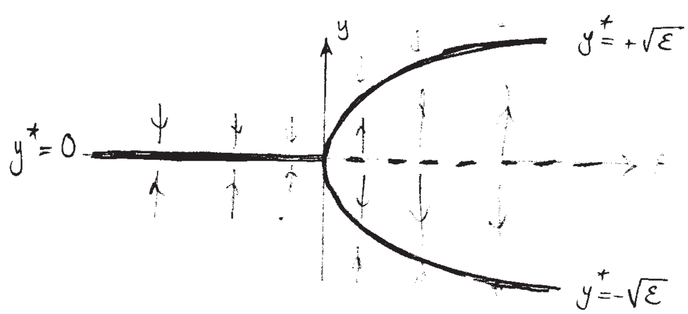

Figure 1.2: Graphical representation of Eq. (1.2.12) with arrows pointing to the direction of $\dot{y}$.

Let us solve for this symmetry breaking by looking at the time evolution $y(t)$ according

[4]An inverted *(subrcritical) pitchfork bifurcation* occurs if the sign of the cubic term in Eq. (1.2.12) is changed to positive, i.e. $\dot{y} = \epsilon y + y^3$. The stable fixed point for $\epsilon < 0$ is now bounded by two unstable fixed points, which merge at $\epsilon = 0$, and the original (now lone) fixed point becomes unstable for $\epsilon > 0$.

to Eq. (1.2.12). Starting from $y(t=0)=y_0$, following the general scheme, we obtain

$$t=\int_{y_0}^{y(t)}\frac{dy'}{y'(\epsilon-y'^2)}=\frac{1}{\epsilon}\int_{y_0}^{y(t)}dy'\left[\frac{1}{y'}+\frac{y'}{\epsilon-y'^2}\right]=\frac{1}{2\epsilon}\int_{y_0}^{y(t)}dy'\frac{d}{dy'}\left[2\ln y'-\ln(\epsilon-y'^2)\right]\,, \tag{1.2.13}$$

which can be rearranged to

$$\frac{y^2}{\epsilon-y^2}=\frac{y_0^2e^{2\epsilon t}}{\epsilon-y_0^2},\quad\Longrightarrow\quad\frac{y^2}{\epsilon}=\frac{y_0^2e^{2\epsilon t}}{\epsilon+y_0^2(e^{2\epsilon t}-1)}\quad\Longrightarrow\quad y(t)=\frac{y_0e^{\epsilon t}}{\sqrt{1+y_0^2(e^{2\epsilon t}-1)/\epsilon}}\,. \tag{1.2.14}$$

Note that in taking the square root in the last step, the plus sign ensures that $\lim_{t\to 0}y(t)=y_0$. For $\epsilon<0$ the solution at long times decays to zero, as $y(t)\propto e^{-|\epsilon|t}$. The behavior for $\epsilon>0$ is more interesting, as

$$\lim_{t\to\infty}y(t)\simeq\frac{y_0e^{\epsilon t}}{\sqrt{y_0^2/\epsilon e^{\epsilon t}}}=\mathrm{sign}(y_0)\sqrt{\epsilon}\,. \tag{1.2.15}$$

The choice between the two symmetry equivalent final steady states is thus made entirely by the location of the starting point. An infinitesimal initial displacement thus determines the outcome of symmetry breaking.