1.2 First order ordinary differential equations

1.2.1 General solution

Equation (**??**) is a typical first order ordinary differential equation (ODE). The stationary points of a general such ODE of the form $\dot{x} = F(x)$ are obtained by as solutions to $F(x^*) = 0$. To plot the trajectory of motion, starting from $x(t = 0) = x_0$, we can use the following procedure[2]

$$\frac{dx}{dt} = F(x), \quad \Longrightarrow \quad \frac{dx}{F(x)} = dt, \quad \Longrightarrow \quad t = \int_{x_0}^{x(t)} \frac{dx'}{F(x')} \,. \tag{1.2.3}$$

Assuming that we can evaluate the integral in the final expression, we still need to invert the result to obtain the explicit form of $x(t)$.

Let us redrive the solution for the decaying linear spring in Eq. (**??**), with $F(x) = -\gamma x$, noting:

$$t = \int_{x_0}^{x(t)} \frac{dx'}{(-\gamma x')} = -\frac{1}{\gamma} \ln\left(\frac{x(t)}{x_0}\right), \quad \Longrightarrow \quad x(t) = x_0 e^{-\gamma t} \,, \tag{1.2.4}$$

as obtained in Eq. (**??**) by summing the Taylor series.

A slight variations is obtained by considering $F(x) = u - \gamma x$, starting from $x = 0$ and $t = 0$, in which case

$$t = \int_{0}^{x(t)} \frac{dx'}{(u - \gamma x')} = -\frac{1}{\gamma} \ln\left(\frac{u - \gamma x(t)}{u}\right), \quad \Longrightarrow \quad x(t) = \frac{u}{\gamma}\left(1 - e^{-\gamma t}\right) \,. \tag{1.2.5}$$

Once again, the coordinate approaches its equilibrium point, $x^* = u/\gamma$ (solution of $F(x^*) = 0$ exponentially. Indeed this solution is identical to the previous one, obtained by a shifting the variable x by $x^* = x_0$.

1.2.2 Population growth – Transcritical bifurcation

After an organism invades a hospitable habitat, its initial rapid reproduction is typically exponential. However, competition for available resources eventually slows down growth. Indicating the size of the population by $N(t)$, and tis rate of change by $G(N)$ leads to the differential equation

$$\frac{dN}{dt} = G(N) = g_0 + g_1 N + g_2 N^2 + \cdots \,. \tag{1.2.6}$$

[2]To justify the first step in Eq. (**??**), recall that from the Taylor series

$$x(t + dt) = x(t) + \frac{dx}{dt} dt + \text{terms of order } dt^2 \text{ and higher.} \tag{1.2.1}$$

Thus to lowest order in dt as $dt \to 0$,

$$dx \equiv x(t + dt) - x(t) = dt\, \frac{dx}{dt} \quad \Longrightarrow \quad dt = \frac{dx}{dx/dt} \,. \tag{1.2.2}$$

The terms in the Taylor expansion of $G(N)$ can now be constrained as applicable to description of a population:

- $g_0 = 0$ if the only mechanism for population growth is reproduction of its existing members.

- $g_1 \equiv r > 0$ is the reproduction rate that by itself leads to exponential growth. In principle, this parameter can be made negative, e.g. by removing needed resources.

- $g_2 < 0$, since competition for resources reduces the reproduction rate. With a view to later interpretation we relabel $g_2 = -r/N^*$.

Typically, the Taylor expansion of $G(N)$ is stopped at second order, resulting in

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N^*}\right) , \tag{1.2.7}$$

with N^* denoting the capacity of the habitat. Following the general procedure, the solution to this ODE can be obtained as follows

$$t = \int_{N_0}^{N(t)} \frac{dN'}{rN'(1 - N'/N^*)} = \frac{1}{r}\int_{N_0}^{N(t)} dN' \left[\frac{1}{N'} + \frac{1}{N^* - N'}\right] . \tag{1.2.8}$$

Note that we can always turn a composite fraction into a sum of fractions involving its factors. this enables easy computation of the integrals, leading to

$$t = \frac{1}{r}\ln\frac{N(t)(N^* - N_0)}{N_0(N^* - N(t))} \qquad \Longrightarrow \qquad \frac{N(t)}{(N^* - N(t))} = \frac{N_0}{(N^* - N_0)}e^{rt} . \tag{1.2.9}$$

An easy way to solve the final equation is by noting that any fraction $A/B = C/D$ can be recast as $A/(A+B) = C/(C+D)$. Applying this to the above equation leads to

$$\frac{N(t)}{N^*} = \frac{N_0 e^{rt}}{N^* + N_0(e^{rt} - 1)} \qquad \Longrightarrow \qquad N(t) = \frac{N^* N_0 e^{rt}}{N^* + N_0(e^{rt} - 1)} . \tag{1.2.10}$$

In the context of population evolution, Eq. (**??**) is known as the *logistic growth*; its solution in Eq. (**??**) is the *logistic growth law*.

Now consider initial populations of the same size N_0 introduced in a range of habitats differing in the parameter r in Eq. (**??**); hospitable habitats correspond to $r > 0$, inhospitable ones to $r < 0$. In considering both positive and negative r it is more natural to write the growth equation as $\dot{N} = rN - g_2 N^2$. For the same value of g_2 (degree of competition), as $r \to 0$, the steady state (long-time) size of the population also vanishes as $N^* = r/g_2$. Clearly, for $r < 0$, a negative solution for N^* is meaningless, and the population collapses. Upon changing the variable r, the function $N^*(r)$ thus changes from $N^* = 0$ for $r < 0$ to $N^* = r/g_2$ for $r > 0$. Such a function that cannot be represented by a single Taylor series around the point $r = 0$, is said to be *non-analytic* at that point. This particular non-analyticity of $N^*(r)$ is an example of a so-called *bifurcation*. It is best understood by noting that Eq. (??) represents gradient descent in the "potential" $V(N) = -rN^2/2 + g_2N^3/3$. The potential has two extrema at $N^* = 0$ and $N^* = r/g_2$. For $r < 0$, the solution at $N^* = 0$ is stable, while for $r > 0$, the one at $N^* = r/g_2$; at $r = 0$, the two extrema collide and change stability.[3]

The above scenario, generic to ODEs of the form

$$\dot{y} = \epsilon y - y^2\,. \tag{1.2.11}$$

is known as a *transcritical* bifurcation. Note that the Logistic Eq. (??) is equivalent to Eq.(??) upon rescaling $N = N^*y/r$ and setting $\epsilon = r$.

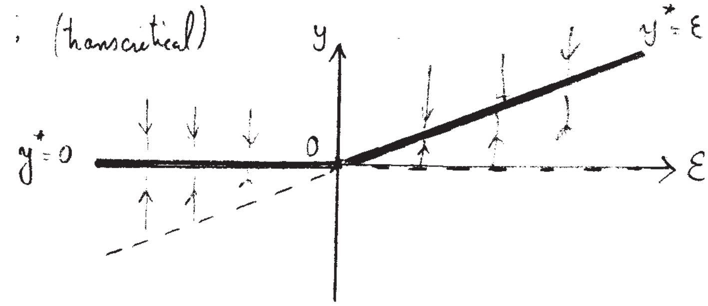

Figure 1.1: Graphical representation of Eq. (??) with arrows pointing to the direction of $\dot{y}$.

1.2.3 Symmetry breaking – Pitchfork bifurcation

In a *generic* Taylor expansion, all terms in the series must be present, unless excluded for a specific reason. For example, the zeroth order term in Eq. (??) is not present in Eq. (??) given the requirement that population growth comes only from reproduction of existing members. The linear term in this equation should generically be present, although its sign can in principle be modified by changing external conditions. The bifurcation point at which $g_1 = 0$ is a prominent example of a special non-generic point in the space of model parameters.

Similarly, the quadratic term in Eq. (??) is generically present, unless forbidden by a fundamental requirement. In many cases, symmetry considerations constrain mathematical descriptions. In particular, for a system with the reflection $y \to -y$, only odd powers of y can appear in the equation for $\dot{y}$, in which case the low order Taylor expansion in Eq. (??) has to be replaced with

$$\dot{y} = \epsilon y - y^3\,. \tag{1.2.12}$$

This equation describes gradient descent in a potential $V(y) = -\epsilon y^2/2 + y^4/3$. For $\epsilon < 0$ the only minimum (stationary) point is at $y^* = 0$, while for $\epsilon > 0$, $y^* = 0$ becomes a maximum

[3]The solutions to $dV/dx|_{x^*} = 0$ correspond to extrema of the function $V(x)$, including both minima and maxima. In a dynamics represented by gradient descent in $V(x)$, the minima represent *stable* end-points while the maxima are *unstable* points. Any small perturbation around a maximum will cause the coordinate to descent to the closest minimum.

(unstable) and a pair of stable fixed points appear at $\pm\sqrt{\epsilon}$. The transition at $\epsilon = 0$ is a *(supercritical) pitchfork bifurcation*, and the choice of one or the other stable point is by *spontaneous symmetry breaking*. Symmetry breaking is a very important concept in physics, occurring in all realms from elementary particles to condensed matter physics (for example ferromagnetism) to cosmology.[4]

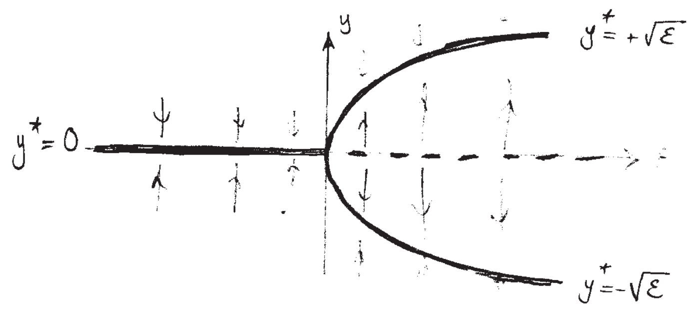

Figure 1.2: Graphical representation of Eq. (**??**) with arrows pointing to the direction of $\dot{y}$.

Let us solve for this symmetry breaking by looking at the time evolution $y(t)$ according to Eq. (**??**). Starting from $y(t = 0) = y_0$, following the general scheme, we obtain

$$t = \int_{y_0}^{y(t)} \frac{dy'}{y'(\epsilon - y'^2)} = \frac{1}{\epsilon}\int_{y_0}^{y(t)} dy' \left[\frac{1}{y'} + \frac{y'}{\epsilon - y'^2}\right] = \frac{1}{2\epsilon}\int_{y_0}^{y(t)} dy' \frac{d}{dy'}\left[2\ln y' - \ln(\epsilon - y'^2)\right] , \tag{1.2.13}$$

which can be rearranged to

$$\frac{y^2}{\epsilon - y^2} = \frac{y_0^2 e^{2\epsilon t}}{\epsilon - y_0^2}, \quad \Longrightarrow \quad \frac{y^2}{\epsilon} = \frac{y_0^2 e^{2\epsilon t}}{\epsilon + y_0^2(e^{2\epsilon t} - 1)} \quad \Longrightarrow \quad y(t) = \frac{y_0 e^{\epsilon t}}{\sqrt{1 + y_0^2(e^{2\epsilon t} - 1)/\epsilon}} . \tag{1.2.14}$$

Note that in taking the square root in the last step, the plus sign ensures that $\lim_{t\to 0} y(t) = y_0$. For $\epsilon < 0$ the solution at long times decays to zero, as $y(t) \propto e^{-|\epsilon|t}$. The behavior for $\epsilon > 0$ is more interesting, as

$$\lim_{t\to\infty} y(t) \simeq \frac{y_0 e^{\epsilon t}}{\sqrt{y_0^2/\epsilon e^{\epsilon t}}} = \text{sign}(y_0)\sqrt{\epsilon} . \tag{1.2.15}$$

The choice between the two symmetry equivalent final steady states is thus made entirely by the location of the starting point. An infinitesimal initial displacement thus determines the outcome of symmetry breaking.

[4] An inverted *(subrcritical) pitchfork bifurcation* occurs if the sign of the cubic term in Eq. (**??**) is changed to positive, i.e. $\dot{y} = \epsilon y + y^3$. The stable fixed point for $\epsilon < 0$ is now bounded by two unstable fixed points, which merge at $\epsilon = 0$, and the original (now lone) fixed point becomes unstable for $\epsilon > 0$.

1.2.4 Universality and critical slowing down

There are many examples in nature when the behavior of a system undergoes a drastic change. Transitions between different phases of matter are a prime example from physics. The bifurcations described above provide a mathematical model for such phenomena. The concept of *universality* captures the applicability of the same mathematical formalism to diverse phenomena. As indication of the power of this concept, note that in both Eq. (**??**) and Eq. (**??**) the time dependence is controlled by an exponential, with a characteristic time scale $\tau = 1/|r|$, or $\tau = 1/|\epsilon|$ that diverges at the transition point. The slow-down of relaxation near points of extinction or symmetry breaking is an important characteristic of these phenomena. We have thus found an underlying mathematical reason for this observation which is completely independent of various details of the process!

At the point $\epsilon = 0$, the exponential dependence gives way to a power-law decay, as

$$\frac{dy}{dt} = -y^p, \quad \Longrightarrow \quad t = \frac{p-1}{y_0^{p-1}} - \frac{p-1}{y^{p-1}}, \quad \Longrightarrow \quad \lim_{t\to\infty} y(t) \propto t^{\frac{1}{p-1}}\,, \tag{1.2.16}$$

with $p = 2$ or $p = 3$ for transcritical or pitchfork bifurcations. You should convince yourself that Eq. (**??**) and Eq. (**??**) indeed exhibit such power-law decays at their transition points.

Recap

- *(i)* Any first order ODE can be solved by

$$\frac{dx}{dt} = F(x), \quad \Longrightarrow \quad \frac{dx}{F(x)} = dt, \quad \Longrightarrow \quad t = \int_{x_0}^{x(t)} \frac{dx'}{F(x')}\,. \tag{1.2.17}$$

- *(ii)* Transcritical and pitchfork bifurcations are captured by the ODEs

$$\dot{y} = \epsilon y - y^p\,, \tag{1.2.18}$$

  with $p = 2$ and $p = 3$ respectively; the latter corresponding to *symmetry breaking.*