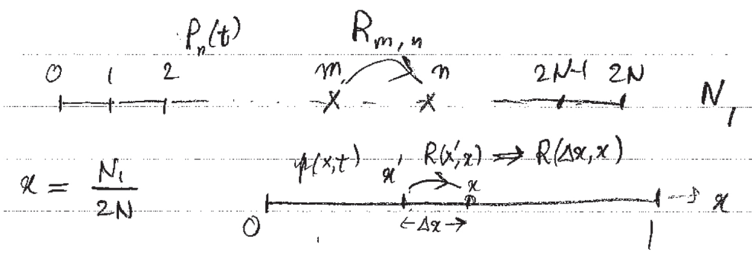
5.2 Continuum limit

5.2.1 Drift and diffusion

Let us now consider evolving probabilities for a generic situation where the states are ordered along a line, such as in the previous examples with population size $n = 0, 1, 2 \cdots, N$. The general form of the Master equation is

$$\frac{dp_n}{dt} = +\sum_{m \neq n} R_{nm} p_m - \sum_{m \neq n} R_{mn} p_n \,. \tag{5.2.1}$$

In many relevant circumstances the number of states is large, and the probability varies smoothly from one value of n to the next. In such cases it is reasonable to replace the discrete index n with a continuous variable x, the probabilities $p_n(t)$ with a probability density $p(x,t)$, and the rates R_{mn} with a rate function $R(x', x)$. The rate function R depends on two variables x and x', denoting respectively the start and end positions for a transition along the line. We have the option of redefining the two arguments of this function, and it is useful to reparameterize it as $\tilde{R}(x' - x, x) \equiv R(x', x)$ indicating the rate at which, starting from the position x, a transition is made to a position $\Delta x = x' - x$ away. As in the case of mutations, there is usually a preference for changes that are *local*, i.e. with rates that decay rapidly when the separation $x' - x$ becomes large.



These transformations and relabelings,

$$n \to x, \;\; p_n(t) \to p(x,t), \;\; R_{mn} \to \tilde{R}(x' - x, x)\,, \tag{5.2.2}$$

enable us to transform Eq. (5.2.1) to the continuous integral equation

$$\frac{\partial}{\partial t} p(x,t) = +\int^{*} dx' \tilde{R}(x - x', x') p(x', t) - \int^{*} dx' \tilde{R}(x' - x, x) p(x,t)\,. \tag{5.2.3}$$

Some care is necessary in replacing the sums with integrals, as the summations in Eq. (5.2.1) exclude the term with $m = n$. To treat this restriction in the continuum limit, we focus on an interval y around any point x, and consider the change in probability due to incoming flux from $x - y$ and the outgoing flux to $x + y$, thus arriving at[3]

$$\frac{\partial}{\partial t} p(x,t) = \int dy \left[\tilde{R}(y, x - y) p(x - y) - \tilde{R}(y, x) p(x) \right]\,. \tag{5.2.4}$$

[3] In Eq. (5.2.3) this amounts to change of variable from x' to $(x - y)$ in the first integral, and to $(x + y)$ in the second.

Note that the contribution for $y = 0$ is now clearly zero. The flux difference for small y is now estimated by a Taylor expansion of the first term in the square bracket, *but only with respect to the location of the incoming flux*, treating the argument pertaining to the separation of the two points as fixed, i.e.

$$\tilde{R}(y, x-y)p(x-y) = \tilde{R}(y, x)p(x) - y\frac{\partial}{\partial x}\left(\tilde{R}(y, x)p(x)\right) + \frac{y^2}{2}\frac{\partial^2}{\partial x^2}\left(\tilde{R}(y, x)p(x)\right) + \cdots . \quad (5.2.5)$$

While formally correct, the above expansion is useful only in cases where typical values of y are small (i.e. when only almost *local* transitions occur). Keeping terms up to the second order, Eq. (5.2.4) can be rewritten as

$$\frac{\partial}{\partial t}p(x, t) = -\int dy\, y\frac{\partial}{\partial x}(\tilde{R}(y, x)p(x)) + \frac{1}{2}\int dy\, y^2\frac{\partial^2}{\partial x^2}(\tilde{R}(y, x)p(x)). \quad (5.2.6)$$

The integrals over y can be taken inside the derivatives with respect to x,

$$\frac{\partial}{\partial t}p(x, t) = -\frac{\partial}{\partial x}\left[p(x)\left(\int dy\, y\tilde{R}(y, x)\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[p(x)\left(\int dy\, y^2\tilde{R}(y, x)\right)\right], \quad (5.2.7)$$

after which we obtain

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}\left[v(x)\ p(x, t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x)p(x, t)\right]. \quad (5.2.8)$$

We have introduced

$$v(x) \equiv \int dy\, y\tilde{R}(y, x) = \frac{\langle\Delta(x)\rangle}{\Delta t}, \quad (5.2.9)$$

and

$$D(x) \equiv \frac{1}{2}\int dy\, y^2\tilde{R}(y, x) = \frac{1}{2}\frac{\langle\Delta(x)^2\rangle}{\Delta t}. \quad (5.2.10)$$

Equation (5.2.8) is a prototypical description of *drift* and *diffusion* which appears in many contexts. The *drift* term $v(x)$ expresses the rate (velocity) with which transitions change (on average) the position from x. Given the probabilistic nature of the process, there are variations in the rate of change of position captured by the position dependent *diffusion* coefficient $D(x)$.[4] The drift–diffusion equation is known as the *forward Kolmogorov* equation in the context of populations. As a description of random walks it appeared earlier in physics literature as the *Fokker–Planck* equation.

[4]The diffusion coefficient is usually associated with the *variance*, $\left\langle\Delta(x)^2\right\rangle_c \equiv \left\langle\Delta(x)^2\right\rangle - \left\langle\Delta(x)\right\rangle^2$. However, in the limit of $\Delta t \to 0$, the squared mean is of second order in Δt, and can be ignored.