5.2.2 Steady states

While it is usually hard to solve a general drift–diffusion equation as a function of time, it is relatively easy to find the steady state solution to which it settles after a long time. Let us denote the steady-state probability distribution by $p^*(x)$, which by definition must satisfy

$$\frac{\partial p^*(x)}{\partial t} = 0. \tag{5.2.11}$$

Therefore, setting the right-hand side of Eq. (5.2.8) to zero, we get

$$-\frac{\partial}{\partial x}\left[v(x)p^*(x)\right] + \frac{\partial^2}{\partial x^2}\left[D(x)p^*(x)\right] = 0. \tag{5.2.12}$$

The most general solution admits steady states in which there is an overall current and the integral over x of the last equation leads to a constant flow in probability. It is not clear how such a circumstance may arise in the context of population genetics, and we shall therefore focus on circumstances where there is no probability current, such that

$$-v(x)p^*(x) + \frac{\partial}{\partial x}(D(x)p^*(x)) = 0. \tag{5.2.13}$$

We can easily rearrange this equation to

$$\frac{1}{D(x)p^*}\frac{\partial}{\partial x}(D(x)p^*(x)) = \frac{\partial}{\partial x}\ln\left(D(x)p^*(x)\right) = \frac{v(x)}{D(x)}\,. \tag{5.2.14}$$

This equation can be integrated to

$$\ln D(x)p^*(x) = \int^x dx'\frac{v(x')}{D(x')} + \text{constant}, \tag{5.2.15}$$

such that

$$p^*(x) \propto \frac{1}{D(x)}\exp\left[\int^x \frac{v(x')}{D(x')}\right], \tag{5.2.16}$$

with the proportionality constant set by boundary conditions.