5.3 Brownian motion

5.3.1 Langevin description

Observations under a microscope indicate that a dust particle in a liquid drop undergoes a random jittery motion. This is because of the random impacts of the much smaller fluid particles. The theory of such (*Brownian*) motion was developed by Einstein in 1905 and starts with the equation of motion for the particle[6]. The displacement $\vec{x}(t)$, of a particle of mass m is governed by,

$$m\,\ddot{\vec{x}} = -\frac{\dot{\vec{x}}}{\mu} - \frac{\partial \mathcal{V}}{\partial \vec{x}} + \vec{f}_{\rm random}(t)\,. \tag{5.3.1}$$

The three forces acting on the particle are:

- A friction force due to the viscosity of the fluid. For a spherical particle of radius R, the mobility in the low Reynolds number limit can be calculated, and is given by $\mu = (6\pi\bar{\eta}R)^{-1}$, where $\bar{\eta}$ is the specific viscosity of the fluid.

- The force due to the external potential $\mathcal{V}(\vec{x})$, e.g. gravity.

- A random force of zero mean due to the impacts of fluid particles.

The viscous term usually dominates the inertial one (i.e. the motion is overdamped), and we shall henceforth ignore the acceleration term. Equation (5.3.1) now reduces to the so called *Langevin equation*,

$$\dot{\vec{x}} = \mu\vec{F}(\vec{x}) + \vec{\eta}(t)\,, \tag{5.3.2}$$

which is the noisy counterpart to Eq. (1.1.22) introduced at the beginning of this course. In addition to the *deterministic* velocity $\vec{v}(\vec{x}) = \mu\vec{F}(\vec{x}) = -\mu\partial\mathcal{V}/\partial\vec{x}$, there is a *stochastic* velocity, $\vec{\eta}(t) = \mu\vec{f}_{\rm random}(t)$; a random variable of zero mean,

$$\langle\vec{\eta}(t)\rangle = 0\,. \tag{5.3.3}$$

Since it represents the net effect of random collisions of a large number of molecules on the immersed particle, we can appeal to the central limit theorem, and assume that its (joint) probability distribution is Gaussian, i.e.

$$p\left[\vec{\eta}(t)\right] \propto \exp\left[-\int d\tau\,\frac{\vec{\eta}(\tau)\cdot\vec{\eta}(\tau)}{4D}\right]\,. \tag{5.3.4}$$

This form implies that different components of the noise, and at different times, are independent, with the covariance

$$\langle\eta_\alpha(t)\eta_\beta(t')\rangle = 2D\delta_{\alpha\beta}\delta(t-t')\,. \tag{5.3.5}$$

[6] A. Einstein, Ann. d. Physik **17**, 549 (1905).

The parameter D is related to *diffusion* of particles in the fluid. In the absence of any potential, $\mathcal{V}(\vec{x}) = 0$, the position of a particle at time t is given by

$$\vec{x}(t) = \vec{x}(0) + \int_0^t d\tau \vec{\eta}(\tau).$$

Clearly the separation $\vec{x}(t) - \vec{x}(0)$ which is the sum of random Gaussian variables is itself Gaussian distributed with mean zero, and a variance

$$\left\langle (\vec{x}(t) - \vec{x}(0))^2 \right\rangle = \int_0^t d\tau_1 d\tau_2 \left\langle \vec{\eta}(\tau_1) \cdot \vec{\eta}(\tau_2) \right\rangle = 3 \times 2Dt.$$

For an ensemble of particles released at $\vec{x}(t) = 0$, i.e. with $\mathcal{P}\left(\vec{x}, t = 0\right) = \delta^3(\vec{x})$, the particles at time t are distributed according to

$$p\left(\vec{x}, t\right) = \left(\frac{1}{\sqrt{4\pi Dt}}\right)^{3/2} \exp\left[-\frac{\vec{x} \cdot \vec{x}}{4Dt}\right],$$

which is the solution to the diffusion equation

$$\frac{\partial p}{\partial t} = D\nabla^2 p\,.$$