5.3.2 Fokker–Planck description

After a small time interval Δt, the position of the particle changes by a random amount given by Eq. (5.3.2) as

$$\vec{\Delta}\vec{x} = \vec{x}(t+\Delta t) - \vec{x}(t) = \mu \vec{F}(\vec{x})\Delta t + \vec{\eta}(t)\Delta t\,. \tag{5.3.6}$$

The procedure outlined in Sec. (5.2.1) can now be used to convert the stochastic Langevin equation into a drift-diffusion equation for the joint PDF $p\,(\vec{x},t)$. Since the random variable $\vec{x}$ has three components, Eq. (5.2.8) is generalized to (using summation convention)

$$\frac{\partial p(\vec{x},t)}{\partial t} = -\frac{\partial}{\partial x_\alpha}\left[v_\alpha(\vec{x})\ p(\vec{x},t)\right] + \frac{\partial^2}{\partial x_\alpha x_\beta}\left[D_{\alpha\beta}(\vec{x})p(\vec{x},t)\right]\,. \tag{5.3.7}$$

Following Eq. (5.2.9) , the drift velocity vector is

$$\vec{v}(\vec{x}) = \frac{\left\langle \vec{\Delta}(\vec{x})\right\rangle}{\Delta t} = \mu\vec{F}(\vec{x})\,, \tag{5.3.8}$$

while generalizing Eq. (5.2.10) components of the diffusion tensor are given by

$$D_{\alpha\beta}(\vec{x}) = \frac{1}{2}\frac{\langle \Delta_\alpha(\vec{x})\Delta_\beta(\vec{x})\rangle}{\Delta t} = \frac{1}{2\Delta t}\left[(\mu F_\alpha \Delta t)\,(\mu F_\beta \Delta t) + (2D\delta_{\alpha\beta}\Delta t)\right] \simeq D\delta_{\alpha\beta} \tag{5.3.9}$$

as $\Delta t \to 0$.

The PDF therefore evolves according to the continuity equation

$$\frac{\partial p}{\partial t} + \nabla\vec{J} = 0 \quad \text{with} \quad \vec{J} = \mu\vec{F}p - D\nabla p\,. \tag{5.3.10}$$

While a free particle diffuses in space over time, in the presence of a confining potential $V(\vec{x})$, it settles to a steady state, with PDF obtained as in in Eq. (5.2.24), as by setting the probability current $\vec{J}$ to zero, yielding

$$\ln p^*(\vec{x}) = \frac{\mu}{D}\int dx'_\alpha F_\alpha(\vec{x}') \quad \Longrightarrow \quad p^*(\vec{x}) \propto \exp\left[-\frac{\mu}{D}V(\vec{x})\right]\,. \tag{5.3.11}$$