1.3 Second order ordinary differential equations

1.3.1 General solution

We observed in Eq. (1.1.18) that the solution to the second order differential equation, $\ddot{x} = -\omega_0^2$, describing simple harmonic oscillations, depends on two parameters x_0 and v_0. It is easy to see from the series method we employed for this solution that the general solution of an n^{th} order ODE will depend on n parameters. However, the solution need not be parametrized as in Eq. (1.1.18) in terms of the initial position and velocity (or higher derivatives). An important example is provided by considering the more general ODE

$$\ddot{x} = F(x)\,. \tag{1.3.1}$$

As described in connection with Eq. (1.1.6), this equation can be integrated once (after multiplication by $\dot{x}$) to give

$$\frac{\dot{x}^2}{2} + V(x) = E, \qquad \text{with} \qquad V(x) = -\int^x dx' F(x')\,. \tag{1.3.2}$$

We have not explicitly indicated a lower cutoff on the integral defining $V(x)$. Changing this lower cutoff modifies $V(x)$ by a constant that can be absorbed in the integration parameter $E = V(x_0) + v_0^2/2$ for $v_0 = \dot{x}(t = 0)$.

Equation (1.3.2) can now be recast into a linear first order ODE and solved by the general method presented earlier, as

$$\frac{dx}{dt} = \pm\sqrt{2(E - V(x))}, \quad \Longrightarrow \quad \pm t = \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{2(E - V(x'))}}\,, \tag{1.3.3}$$

with the choice of sign dictated by relevant considerations, e.g. the sign of the initial velocity $v_0 = \dot{x}(t = 0)$.