### 1.3.2 Complex exponentials and SHO

In Eq. (1.1.18) the general solution for simple harmonic oscillations was expressed as a sum of trigonomic functions, sine and cosine. In the same way that a simple exponential $e^t$ is the solution of the simplest linear ODE $\dot{x} = x$, sine and cosine are natural solutions of $\ddot{x} = -x$. Knowing any number of derivatives still convert an exponential to itself, let us try out the exponential $x_0 e^{\lambda t}$ as a solution to the equation, $\ddot{x} + \omega_0^2 x = 0$. Taking two derivatives yields

$$\lambda^2 x_0 e^{\lambda t} = -\omega_0^2 x_0 e^{\lambda t}, \quad \Rightarrow \quad \lambda = \pm i\omega_0, \tag{1.3.4}$$

where $\omega_0$ is a real number, while $i$ stands for the square root of -1 ($i = \sqrt{-1}$), such that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, etc. Note that some texts use $j$ to denote $\sqrt{-1}$, which can be confusing. It is easily checked that the sum of two such exponentials is also a solution, and thus the most general solution can thus be written as

$$x(t) = c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t},$$

where $c_+$ and $c_-$ can in fact be complex numbers. This may appear strange, since each complex number, $z = a + ib$, has both a real and imaginary part, giving the impression that the general solution depends on 4 independent parameters. The resolution is that for classical[5] physical problems we are only interested in real solutions to the problem, and must choose $c_+ = c_-^* = c$,[6] and the solution is then given by the *real part* as

$$x(t) = \Re\left[c e^{i\omega_0 t}\right].$$

To gain better understanding of the complex exponential, let us examine its power series

$$\begin{aligned} e^{i\omega_0 t} &= 1 + i\omega_0 t + \frac{(i\omega_0 t)^2}{2!} + \frac{(i\omega_0 t)^3}{3!} + \frac{(i\omega_0 t)^4}{4!} + \cdots \\ &= \left[1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - \cdots\right] + i\left[\omega_0 t - \frac{(\omega_0 t)^3}{3!} + \cdots\right] \\ &= \cos(\omega_0 t) + i\sin(\omega_0 t), \end{aligned} \tag{1.3.5}$$

where we have used the power series for sine and cosine. This is the famous *Euler formula.* From the the above equation, and its complex conjugate

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t), \quad e^{-i\omega_0 t} = \cos(\omega_0 t) - i\sin(\omega_0 t),$$

we obtain

$$\cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}, \quad \sin(\omega_0 t) = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}.$$

[5] While complex functions play an important role in quantum description of physical phenomena, their consideration at this point is not needed.

[6] The star superscript denotes complex conjugation, as in $(2 + 3i)^* = 2 - 3i$.

Hence our original solution can be written as

$$\begin{aligned} x(t) &= x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \\ &= x_0 \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \frac{v_0}{\omega_0} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \\ &= \Re \left[ \left( x_0 - i \frac{v_0}{\omega_0} \right) e^{i\omega_0 t} \right]. \end{aligned} \tag{1.3.6}$$