1.3.3 Geometric representation

A complex number $z = x + iy$ can be represented by a point with *cartesian coordinates* (x, y) in complex plane. The point can also be described by *polar coordinates* (r, ϕ). Consider the line from the the origin to the point (x, y). This line has length $r = \sqrt{x^2 + y^2}$ (Pythagorean theorem), and makes angle ϕ to the x axis. The geometric definitions of trigonometric functions imply $\sin\phi = y/r$, $\cos\phi = x/r$, $\tan\phi = y/x$, and $\cot\phi = x/y$. Thus the relations between the two sets of coordinates are

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}, \qquad \Longleftrightarrow \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \end{cases}. \tag{1.3.7}$$

Applying these rules to $e^{i\omega_0 t}$, we see that this complex number has *magnitude* $r = 1$, and is at the polar angle $\phi = \omega_0 t$. As a function of time, this corresponds to a point that rotates on a unit circle in the complex plane with angular velocity ω_0.

Any complex number c can thus be written in terms of two real parameters in two ways, as

$$c = c_1 + ic_2, \qquad \text{or} \qquad c = Ae^{i\phi_0}.$$

Using the second form, the general solution to the SHO can also be written as

$$x(t) = \Re\left[ce^{i\omega_0 t}\right] = A\cos\left(\omega_0 t + \phi_0\right),$$

where A is the *amplitude* and ϕ_0 is the *phase*. What is the amplitude and phase of the SHO solution we derived initially with parameters x_0 and v_0? In this case, the complex amplitude is

$$c = \left(x_0 - i\frac{v_0}{\omega_0}\right),$$

whose magnitude and phase are given by

$$A = \sqrt{x_0^2 + v_0^2/\omega_0^2}, \qquad \phi = -\tan^{-1}\left(\frac{v_0}{x_0\omega_0}\right).$$