1.3.4 Addition of complex numbers & beats

Complex exponentials are very useful for proving trigonometric identities. For example, noting that $e^{ia} \times e^{ib} = e^{i(a+b)}$, and employing $e^{ia} = \cos a + i \sin a$, leads to

$$\begin{aligned} e^{ia} \times e^{ib} &= (\cos a + i \sin a) \times (\cos b + i \sin b) \\ &= (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \cos b \sin a) \\ &= e^{i(a+b)} = \cos(a+b) + i \sin(a+b). \end{aligned} \tag{1.3.8}$$

Comparing the real and imaginary parts on second and third rows of above equation, we get the identities

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \text{and} \quad \sin(a+b) = \cos a \sin b + \sin a \cos b. \tag{1.3.9}$$

Noting that cosine is even in its argument, while sine is odd, changing the sign of b in the above equation leads to

$$\cos(a-b) = \cos a \cos b + \sin a \sin b, \quad \text{and} \quad \sin(a-b) = -\cos a \sin b + \sin a \cos b. \tag{1.3.10}$$

Finally, adding the two sets of equations leads to the identities

$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b, \quad \text{and} \quad \sin(a+b) + \sin(a-b) = 2 \sin a \cos b. \tag{1.3.11}$$

In many physical situations SHO signals are superposed on passing through a medium, such as in electromagnetic waves traversing vacuum, or sound waves going through air. An interesting example is provided by the sound of two tuning forks with slightly different frequencies. In addition to the tone for the average frequency, the ear hears an alternating beating at a much lower frequency. Assuming that the amplitudes coming from the two tuning forks are the same, the total signal can be constructed as

$$s(t) = A \left[\cos(\omega_1 t + \phi_1) + \cos(\omega_2 t + \phi_2) \right].$$

Using a version of the above identities above, $\cos a + \cos b = 2 \cos\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$, we find

$$s(t) = 2A \cos\left(\frac{\omega_1 - \omega_2}{2} t + \frac{\phi_1 - \phi_2}{2} \right) \cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{\phi_1 + \phi_2}{2} \right),$$

i.e. the net signal resembles simple harmonic oscillations at the average frequency (the second cosine), but with an amplitude that is modulated at the difference in frequencies (the first cosine).