1.3.5 Hyperbolic trig–functions

While complex exponentials are solutions of the ODE $\ddot{x} = -\omega_0^2 x$, solutions to $\ddot{x} = \lambda^2 x$ are sums of real exponentials proportional to $e^{\lambda t}$ and $e^{-\lambda t}$. As in Eq. (1.3.5), we can rewrite the sum in terms of hyperbolic sine and cosine, defined by

$$\cosh(\lambda t) = \frac{e^{\lambda t} + e^{-\lambda t}}{2}, \quad \text{and} \quad \sinh(\lambda t) = \frac{e^{\lambda t} - e^{-\lambda t}}{2}. \tag{1.3.12}$$

The advantage of the above combinations is that (like their trigonometric counterparts) they are respectively *symmetric* and *anti-symmetric*, under change of sign of their arguments. Finding solutions that respect underlying symmetries of a physical problem is always important and something that we shall return to repeatedly.

The hyperbolic analogs of tangent (and cotangent) are similarly constructed as

$$\tanh a = \frac{\sinh a}{\cosh a} = \frac{e^a - e^{-a}}{e^a + e^{-a}} = \frac{e^{2a} - 1}{e^{2a} + 1}, \quad \text{and} \quad \coth a = \frac{1}{\tanh a}. \tag{1.3.13}$$

Note the identities

$$\cosh^2 a - \sinh^2 a = 1, \quad \Longrightarrow \quad 1 - \tanh^2 a = \frac{1}{\cosh^2 a}. \tag{1.3.14}$$

While sine (sinh) and cosine (cosh) are solutions to $\ddot{x} = -a^2 x$ ($\ddot{x} = +a^2 x$), it is instructive to learn ODEs satisfies by tangent functions. Towards this end, note the following

$$x = \tan t = \frac{\sin t}{\cos t}, \quad \Rightarrow \quad \dot{x} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = 1 + x^2, \tag{1.3.15}$$

and

$$x = \tanh t = \frac{\sinh t}{\cosh t}, \quad \Rightarrow \quad \dot{x} = \frac{\cosh^2 t - \sin^2 t}{\cosh^2 t} = 1 - x^2, \tag{1.3.16}$$

Equations (1.3.15) and (1.3.16) are indicative of the type of (nonlinear) first order ODE whose solution can be expressed in terms of tangent functions.

Corresponding second order ODE's are obtained by taking another derivative, as

$$x = \tan t, \quad \Rightarrow \quad \dot{x} = 1 + x^2, \quad \Rightarrow \quad \ddot{x} = 2x\dot{x} = 2x(1 + x^2), \tag{1.3.17}$$

and

$$x = \tanh t, \quad \Rightarrow \quad \dot{x} = 1 - x^2, \quad \Rightarrow \quad \ddot{x} = -2x\dot{x} = -2x(1 - x^2). \tag{1.3.18}$$

Equation $\ddot{x} = 2x\dot{x} = 2x(1 - x^2)$ is particularly interesting as it describes motion of a particle (of unit mass) in a potential $V(x) = +x^2 - x^4/2$. This is a symmetric potential with maxima at $x = \pm 1$. The solution $x(t) = \tanh t$ describes a particle that starts close to $x = -1$ at long times in the past ($t \to -\infty$), slides down to reach the minimum at $x = 0$ at $t = 0$, and then climbs the barrier towards $x = +1$, with just enough energy to reach this maximum at $t \to \infty$. Such solutions connecting one extremum to another appear in various physics contexts, as *solitons* in field theory, or *domain walls* in magnets.