1.3 Second order ordinary differential equations

1.3.1 General solution

We observed in Eq. (1.1.18) that the solution to the second order differential equation, $\ddot{x} = -\omega_0^2$, describing simple harmonic oscillations, depends on two parameters x_0 and v_0. It is easy to see from the series method we employed for this solution that the general solution of an n^{th} order ODE will depend on n parameters. However, the solution need not be parametrized as in Eq. (1.1.18) in terms of the initial position and velocity (or higher derivatives). An important example is provided by considering the more general ODE

$$\ddot{x} = F(x)\,. \tag{1.3.1}$$

As described in connection with Eq. (1.1.6), this equation can be integrated once (after multiplication by $\dot{x}$) to give

$$\frac{\dot{x}^2}{2} + V(x) = E, \qquad \text{with} \qquad V(x) = -\int^x dx' F(x')\,. \tag{1.3.2}$$

We have not explicitly indicated a lower cutoff on the integral defining $V(x)$. Changing this lower cutoff modifies $V(x)$ by a constant that can be absorbed in the integration parameter $E = V(x_0) + v_0^2/2$ for $v_0 = \dot{x}(t = 0)$.

Equation (1.3.2) can now be recast into a linear first order ODE and solved by the general method presented earlier, as

$$\frac{dx}{dt} = \pm\sqrt{2(E - V(x))}, \quad \Longrightarrow \quad \pm t = \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{2(E - V(x'))}}\,, \tag{1.3.3}$$

with the choice of sign dictated by relevant considerations, e.g. the sign of the initial velocity $v_0 = \dot{x}(t = 0)$.

1.3.2 Complex exponentials and SHO

In Eq. (1.1.18) the general solution for simple harmonic oscillations was expressed as a sum of trigonomic functions, sine and cosine. In the same way that a simple exponential e^t is the solution of the simplest linear ODE $\dot{x} = x$, sine and cosine are natural solutions of $\ddot{x} = -x$. Knowing any number of derivatives still convert an exponential to itself, let us try out the exponential $x_0 e^{\lambda t}$ as a solution to the equation, $\ddot{x} + \omega_0^2 x = 0$. Taking two derivatives yields

$$\lambda^2 x_0 e^{\lambda t} = -\omega_0^2 x_0 e^{\lambda t}, \quad \Rightarrow \quad \lambda = \pm i\omega_0, \tag{1.3.4}$$

where ω_0 is a real number, while i stands for the square root of -1 ($i = \sqrt{-1}$), such that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, etc. Note that some texts use j to denote $\sqrt{-1}$, which can be confusing. It is easily checked that the sum of two such exponentials is also a solution, and thus the most general solution can thus be written as

$$x(t) = c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t},$$

where c_+ and c_- can in fact be complex numbers. This may appear strange, since each complex number, $z = a + ib$, has both a real and imaginary part, giving the impression that the general solution depends on 4 independent parameters. The resolution is that for classical[5] physical problems we are only interested in real solutions to the problem, and must choose $c_+ = c_-^* = c$,[6] and the solution is then given by the *real part* as

$$x(t) = \Re\left[ce^{i\omega_0 t}\right].$$

To gain better understanding of the complex exponential, let us examine its power series

$$\begin{aligned} e^{i\omega_0 t} &= 1 + i\omega_0 t + \frac{(i\omega_0 t)^2}{2!} + \frac{(i\omega_0 t)^3}{3!} + \frac{(i\omega_0 t)^4}{4!} + \cdots \\ &= \left[1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - \cdots\right] + i\left[\omega_0 t - \frac{(\omega_0 t)^3}{3!} + \cdots\right] \\ &= \cos(\omega_0 t) + i\sin(\omega_0 t), \end{aligned} \tag{1.3.5}$$

where we have used the power series for sine and cosine. This is the famous *Euler formula.* From the the above equation, and its complex conjugate

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t), \quad e^{-i\omega_0 t} = \cos(\omega_0 t) - i\sin(\omega_0 t),$$

we obtain

$$\cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}, \quad \sin(\omega_0 t) = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}.$$

Hence our original solution can be written as

$$\begin{aligned} x(t) &= x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) \\ &= x_0\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \frac{v_0}{\omega_0}\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \\ &= \Re\left[\left(x_0 - i\frac{v_0}{\omega_0}\right)e^{i\omega_0 t}\right]. \end{aligned} \tag{1.3.6}$$

1.3.3 Geometric representation

A complex number $z = x + iy$ can be represented by a point with *cartesian coordinates* (x, y) in complex plane. The point can also be described by *polar coordinates* (r, ϕ). Consider the line from the the origin to the point (x, y). This line has length $r = \sqrt{x^2 + y^2}$ (Pythagorean theorem), and makes angle ϕ to the x axis. The geometric definitions of trigonometric functions imply $\sin\phi = y/r$, $\cos\phi = x/r$, $\tan\phi = y/x$, and $\cot\phi = x/y$. Thus the relations between the two sets of coordinates are

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}, \qquad \Longleftrightarrow \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \end{cases}. \tag{1.3.7}$$

[5]While complex functions play an important role in quantum description of physical phenomena, their consideration at this point is not needed.

[6]The star superscript denotes complex conjugation, as in $(2 + 3i)^* = 2 - 3i$.

Applying these rules to $e^{i\omega_0 t}$, we see that this complex number has *magnitude* $r = 1$, and is at the polar angle $\phi = \omega_0 t$. As a function of time, this corresponds to a point that rotates on a unit circle in the complex plane with angular velocity ω_0.

Any complex number c can thus be written in terms of two real parameters in two ways, as

$$c = c_1 + ic_2, \qquad \text{or} \qquad c = Ae^{i\phi_0}.$$

Using the second form, the general solution to the SHO can also be written as

$$x(t) = \Re\left[ce^{i\omega_0 t}\right] = A\cos\left(\omega_0 t + \phi_0\right),$$

where A is the *amplitude* and ϕ_0 is the *phase*. What is the amplitude and phase of the SHO solution we derived initially with parameters x_0 and v_0? In this case, the complex amplitude is

$$c = \left(x_0 - i\frac{v_0}{\omega_0}\right),$$

whose magnitude and phase are given by

$$A = \sqrt{x_0^2 + v_0^2/\omega_0^2}, \qquad \phi = -\tan^{-1}\left(\frac{v_0}{x_0\omega_0}\right).$$

1.3.4 Addition of complex numbers & beats

Complex exponentials are very useful for proving trigonometric identities. For example, noting that $e^{ia} \times e^{ib} = e^{i(a+b)}$, and employing $e^{ia} = \cos a + i\sin a$, leads to

$$\begin{aligned} e^{ia} \times e^{ib} &= (\cos a + i\sin a) \times (\cos b + i\sin b) \\ &= (\cos a\cos b - \sin a\sin b) + i(\cos a\sin b + \cos b\sin a) \\ &= e^{i(a+b)} = \cos(a+b) + i\sin(a+b). \end{aligned} \tag{1.3.8}$$

Comparing the real and imaginary parts on second and third rows of above equation, we get the identities

$$\cos(a+b) = \cos a\cos b - \sin a\sin b, \quad \text{and} \quad \sin(a+b) = \cos a\sin b + \sin a\cos b. \tag{1.3.9}$$

Noting that cosine is even in its argument, while sine is odd, changing the sign of b in the above equation leads to

$$\cos(a-b) = \cos a\cos b + \sin a\sin b, \quad \text{and} \quad \sin(a-b) = -\cos a\sin b + \sin a\cos b. \tag{1.3.10}$$

Finally, adding the two sets of equations leads to the identities

$$\cos(a+b) + \cos(a-b) = 2\cos a\cos b, \quad \text{and} \quad \sin(a+b) + \sin(a-b) = 2\sin a\cos b. \tag{1.3.11}$$

In many physical situations SHO signals are superposed on passing through a medium, such as in electromagnetic waves traversing vacuum, or sound waves going through air.

An interesting example is provided by the sound of two tuning forks with slightly different frequencies. In addition to the tone for the average frequency, the ear hears an alternating beating at a much lower frequency. Assuming that the amplitudes coming from the two tuning forks are the same, the total signal can be constructed as

$$s(t) = A\left[\cos\left(\omega_1 t + \phi_1\right) + \cos\left(\omega_2 t + \phi_2\right)\right].$$

Using a version of the above identities above, $\cos a + \cos b = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$, we find

$$s(t) = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t + \frac{\phi_1 - \phi_2}{2}\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t + \frac{\phi_1 + \phi_2}{2}\right),$$

i.e. the net signal resembles simple harmonic oscillations at the average frequency (the second cosine), but with an amplitude that is modulated at the difference in frequencies (the first cosine).

1.3.5 Hyperbolic trig–functions

While complex exponentials are solutions of the ODE $\ddot{x} = -\omega_0^2 x$, solutions to $\ddot{x} = \lambda^2 x$ are sums of real exponentials proportional to $e^{\lambda t}$ and $e^{-\lambda t}$. As in Eq. (1.3.5), we can rewrite the sum in terms of hyperbolic sine and cosine, defined by

$$\cosh(\lambda t) = \frac{e^{\lambda t} + e^{-\lambda t}}{2}, \quad \text{and} \quad \sinh(\lambda t) = \frac{e^{\lambda t} - e^{-\lambda t}}{2}. \tag{1.3.12}$$

The advantage of the above combinations is that (like their trigonometric counterparts) they are respectively *symmetric* and *anti-symmetric*, under change of sign of their arguments. Finding solutions that respect underlying symmetries of a physical problem is always important and something that we shall return to repeatedly.

The hyperbolic analogs of tangent (and cotangent) are similarly constructed as

$$\tanh a = \frac{\sinh a}{\cosh a} = \frac{e^a - e^{-a}}{e^a + e^{-a}} = \frac{e^{2a} - 1}{e^{2a} + 1}, \quad \text{and} \quad \coth a = \frac{1}{\tanh a}. \tag{1.3.13}$$

Note the identities

$$\cosh^2 a - \sinh^2 a = 1, \quad \Longrightarrow \quad 1 - \tanh^2 a = \frac{1}{\cosh^2 a}. \tag{1.3.14}$$

While sine (sinh) and cosine (cosh) are solutions to $\ddot{x} = -a^2 x$ ($\ddot{x} = +a^2 x$), it is instructive to learn ODEs satisfies by tangent functions. Towards this end, note the following

$$x = \tan t = \frac{\sin t}{\cos t}, \quad \Rightarrow \quad \dot{x} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = 1 + x^2, \tag{1.3.15}$$

and

$$x = \tanh t = \frac{\sinh t}{\cosh t}, \quad \Rightarrow \quad \dot{x} = \frac{\cosh^2 t - \sin^2 t}{\cosh^2 t} = 1 - x^2, \tag{1.3.16}$$

Equations (1.3.15) and (1.3.16) are indicative of the type of (nonlinear) first order ODE whose solution can be expressed in terms of tangent functions.

Corresponding second order ODE's are obtained by taking another derivative, as

$$x = \tan t, \quad \Rightarrow \quad \dot{x} = 1 + x^2, \quad \Rightarrow \quad \ddot{x} = 2x\dot{x} = 2x(1 + x^2)\,, \tag{1.3.17}$$

and

$$x = \tanh t, \quad \Rightarrow \quad \dot{x} = 1 - x^2, \quad \Rightarrow \quad \ddot{x} = -2x\dot{x} = -2x(1 - x^2)\,. \tag{1.3.18}$$

Equation $\ddot{x} = 2x\dot{x} = 2x(1 - x^2)$ is particularly interesting as it describes motion of a particle (of unit mass) in a potential $V(x) = +x^2 - x^4/2$. This is a symmetric potential with maxima at $x = \pm 1$. The solution $x(t) = \tanh t$ describes a particle that starts close to $x = -1$ at long times in the past ($t \to -\infty$), slides down to reach the minimum at $x = 0$ at $t = 0$, and then climbs the barrier towards $x = +1$, with just enough energy to reach this maximum at $t \to \infty$. Such solutions connecting one extremum to another appear in various physics contexts, as *solitons* in field theory, or *domain walls* in magnets.

Recap

- *(i)* Any second order ODE can be solved by using energy as a first integral, as

$$\ddot{x} = F(x), \quad \Longrightarrow \quad \frac{dx}{dt} = \pm\sqrt{2(E - V(x))}, \quad \Longrightarrow \quad \pm t = \int_{x_0}^{x(t)} \frac{dx'}{2(E - V(x'))}\,, \tag{1.3.19}$$

- *(ii)* Complex exponentials, solutions to the linear ODR $\ddot{x} = -\omega_0^2 x$, follow the *Euler relation*

$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t\,, \tag{1.3.20}$$

  which can be used to switch between polar and cartesian representation, and to derive various trigonometric identities.