1.4 General linear ordinary differential equations

1.4.1 General solution

The general form of an mth order, linear, homogeneous[7], differential equation is

$$a_m \frac{d^m x}{dt^m} + a_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0\,, \tag{1.4.1}$$

where $\{a_m, \cdots, a_0\}$ are fixed parameters. Given our past success with the exponential function, we can guess that a particular solution to this equation should be of the form $x(t) = ce^{\lambda t}$. It is then easy to check that each subsequent derivative multiplies $x(t)$ by a factor λ, such that

$$\frac{d^m x}{dt^m} = \lambda^m x(t)\,. \tag{1.4.2}$$

Substituting this result into the differential equation gives

$$\left[a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0\right] x(t) = 0\,. \tag{1.4.3}$$

Allowed values of λ are obtained by solving for where the expression in the square brackets is zero. In fact, this mth order algebraic equation has m solutions, which we shall label $\{\lambda_1, \cdots, \lambda_m\}$. Any of these values gives an acceptable particular solution.

An important property of homogeneous linear ODEs is that a general solution obtained by adding the particular solutions. It is valuable to remember that this *superposition principle* applies to linear systems only, and fails if any non-linearity is present. The most general solution of Eq. (1.4.1) is thus given by

$$x(t) = c_1 e^{\lambda_1 t} + \cdots + c_m e^{\lambda_m t}\,. \tag{1.4.4}$$

As anticipated, this solution depends on m independent parameters $\{c_1, \cdots, c_m\}$. In the description of most physical systems, the parameters $\{a_m, \cdots, a_0\}$ describing the ODE, and hence the coefficients in the polynomial equation

$$a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0 = 0 \tag{1.4.5}$$

are real numbers. This implies that if λ_i is a particular solution of this equation, so is its complex conjugate λ_i^*. Thus solutions to the equation are either real, or appear in complex conjugate pairs $a \pm i\omega$. The latter type of solutions can be combined to provide real solutions (such as $e^{at}\sin(\omega t)$) that oscillate with frequency ω, and grow or decay exponentially in time. The damped harmonic motion discussed next provides an important prototype of such behavior.

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[7] For inhomogeneous equations, to be discussed later, the right hand side is not zero.