1.4.2 General Damped Harmonic Motion

We noted earlier that if variations in time do not change (conserve) the 'energy' $E = \dot{x}^2/2 + V(x)$, the resulting motion is governed by the second order ODE, $\ddot{x} + V'(x) = 0$, which satisfies time reversal symmetry. Damping is, however, present in mechanical systems, causing irreversible loss of energy. Friction, air drag, viscosity, are all forms of energy dissipation, and the resulting energy loss is in principle a complex function of the motion $x(t)$. We again appeal to instantaneity and continuity to postulate that for continuous motion dE/dt is a function of velocity $\dot{x}$, whose series expansion must start with a lowest order term proportional to $-\dot{x}^2$. (A constant term would cause continuous decrease of energy in the absence of motion, while a linear term can lead to increase of energy depending on the sign of $\dot{x}$.) The loss of energy is then expressed as[8]

$$-\gamma\dot{x}^2 = \frac{dE}{dt} = \frac{d}{dt}\left(\frac{\dot{x}^2}{2} + V(x)\right) = \dot{x}\left[\ddot{x} + V'(x)\right], \quad \Rightarrow \quad \ddot{x} = -V'(x) - \gamma\dot{x}. \tag{1.4.6}$$

The last equation could have been also obtained by equating the net force to the acceleration, assuming a friction force $F_v = b\dot{x}$. Small amplitude deformations, approximated as in Eq. (1.1.6), in the presence of damping, are now described by the linear differential equation

$$m\ddot{x} + b\dot{x} + Kx = 0\,, \tag{1.4.7}$$

which can again be brought to the more standard form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\,, \tag{1.4.8}$$

by setting $\gamma = b/m$, and $\omega_0^2 = K/m$.

Following the general scheme, we can seek particular solutions to the above equation by trying the exponential form $x = ce^{\lambda t}$. Since $\dot{x} = \lambda x$ and $\ddot{x} = \lambda^2 x$, λ must be a solution to the quadratic equation

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0\,. \tag{1.4.9}$$

The two solutions to this equation are

$$\lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,. \tag{1.4.10}$$

The character of the solutions changes dramatically depending on whether the quantity under the square–root is positive or negative. This in turn is controlled by the ratio

$$Q = \frac{\omega_0}{\gamma},$$

which is commonly referred to as the *quality factor*. We shall describe the three classes of possible solutions in turn.

[8]Recall that according to the chain rule in Eq. (1.1.23), $\frac{d}{dt}V(x) = \frac{dV}{dx}\frac{dx}{dt}$.