1.4.3 Forms of Damped motion

1. Over-damped motion: For $\gamma > 2\omega_0$ ($Q < 1/2$), the solutions for λ are *real*, and can be written as

$$\lambda_\pm = -\frac{\gamma}{2} \pm s, \qquad \text{where} \qquad s = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2}\sqrt{1-4Q^2}\,. \tag{1.4.11}$$

Note that s is a real number, with λ_+ and λ_- both negative. Each choice of λ thus results in an exponentially damped solution, eventually decaying to zero at large times. For this reason, the over-damped behavior is also called *dead beat*. Naturally, the very long time behavior is controlled by the slower of the two exponential decays corresponding to λ_+.

As a specific example, consider a system launched from the equilibrium point $x = 0$ at time $t = 0$, with the initial velocity $\dot{x}(t = 0) = v_0$. The general solution to the motion is obtained by the superposition of the two exponentials, as

$$\begin{cases} x(t) = c_+ e^{-\frac{\gamma}{2}t+st} + c_- e^{-\frac{\gamma}{2}t-st}, \\ \dot{x}(t) = c_+\left(-\frac{\gamma}{2}+s\right)e^{-\frac{\gamma}{2}t+st} + c_-\left(-\frac{\gamma}{2}-s\right)e^{-\frac{\gamma}{2}t-st}. \end{cases} \tag{1.4.12}$$

From $x(0) = 0$, we get $c_- = -c_+$, while

$$\dot{x}(0) = v_0 = c_+\left[-\frac{\gamma}{2} - s + \frac{\gamma}{2} + s\right] = 2c_+ s, \quad \Rightarrow \quad c_+ = \frac{v_0}{2s}\,. \tag{1.4.13}$$

The full solution is thus given by

$$x(t) = \frac{v_0}{2s}\left(e^{-\frac{\gamma}{2}t+st} - e^{-\frac{\gamma}{2}t-st}\right) = v_0 e^{-\gamma t/2}\frac{\sinh(st)}{s} = v_0 e^{-\gamma t/2}\frac{\sinh\left(\sqrt{\gamma^2/4-\omega_0^2}\,t\right)}{\sqrt{\gamma^2/4-\omega_0^2}}\,. \tag{1.4.14}$$

2. Critical damping: At the special value of $Q = \omega_0/\gamma = 1/2$ the two solutions for λ merge to the single value of $-\gamma/2$. However, we expect that the second order differential equation should have two independent solutions. We can check that $x_2(t) = te^{-\gamma t/2}$ is also a solution:[9]

$$x_2(t) = e^{-\gamma t/2}t, \Rightarrow \quad \dot{x}_2(t) = e^{-\gamma t/2}\left(-\frac{\gamma}{2}t + 1\right), \Rightarrow \quad \ddot{x}_2(t) = e^{-\gamma t/2}\left(\frac{\gamma^2}{4}t - \frac{\gamma}{2} - \frac{\gamma}{2}\right)\,. \tag{1.4.15}$$

Substituting into the original solution, and using $\omega_0^2 = \gamma^2/4$, confirms that x_2 is indeed a solution.

The general solution in this critically damped case has thus the form

$$x(t) = e^{-\gamma t/2}\left(c_1 + c_2 t\right)\,. \tag{1.4.16}$$

[9]Quite generally when two or more exponents of trial exponential solutions to linear differential equations become equal, we can find new solutions by multiplying the exponentials by powers of t.

For a solution that starts at the origin for $t = 0$, we must have $c_1 = 0$. If the initial velocity is v_0, we thus obtain

$$x(t) = v_0 t e^{-\gamma t/2}\,. \tag{1.4.17}$$

This solution goes through a maximum at time $t_{\max} = 2/\gamma$ (from $\dot{x} = 0$), and then decays to the origin. For a fixed γ, as ω_0 is increased from zero, the fastest decay time is obtained at critical damping.

3. Under-damped motion: For $\omega_0 > \gamma/2$ ($Q > 1/2$), the solutions for λ are *complex*, and given by

$$\lambda_\pm = -\frac{\gamma}{2} \pm i\tilde{\omega}, \qquad \text{where} \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}\,. \tag{1.4.18}$$

The general solution to the motion is obtained by the superposition of exponentials, as

$$x(t) = c_+ e^{-\frac{\gamma}{2}t + i\tilde{\omega}t} + c_- e^{-\frac{\gamma}{2}t - i\tilde{\omega}t}.$$

As noted earlier the two exponential solutions are complex conjugates, and a real solution for $x(t)$ can be obtained by setting $c_- = c_+^*$, and written as

$$x(t) = \Re\left[c e^{-\frac{\gamma}{2}t + i\tilde{\omega}t}\right] = \tilde{A} e^{-\gamma t/2}\cos(\tilde{\omega}t + \tilde{\phi})\,, \tag{1.4.19}$$

where in the last step, we have assumed a complex number $c = \tilde{A}e^{i\tilde{\phi}}$, with amplitude $\tilde{A}$, and phase $\tilde{\phi}$.

This solution has the following properties:

- Because of the imaginary exponential, it has an oscillatory character, with a period $T = 2\pi/\tilde{\omega}$. Introduction of damping (finite γ) reduces the frequency from ω_0. The period thus goes up, eventually diverging to infinity as $\gamma \to 2\omega_0$.

- The amplitude of oscillations decays with time as $e^{-\gamma t/2}$. The characteristic decay time is proportional to $1/\gamma$, and is independent of ω_0.

- Over the characteristic decay time $\tau \sim 1/\gamma$, the number of oscillations $N \sim 2\pi/\tilde{\omega}\tau \sim \gamma/\tilde{\omega}$ is roughly set by Q: A high quality oscillator executes many oscillations before dying away, while a low quality one has few oscillations.

The phase of the solution depends on the initial conditions. For example, starting with $x(t = 0) = 0$ and $\dot{x}(t = 0) = v_0$ gives

$$x(t) = v_0 e^{-\gamma t/2}\frac{\sin(\tilde{\omega}t)}{\tilde{\omega}}\,. \tag{1.4.20}$$

(This solution is the *analytic continuation* of that obtained in the over-damped case, with $s \to i\tilde{\omega}$.)