1.4.4 Inhomogeneous linear ODEs

The *inhomogeneous* variant of the nth order, linear equation in Eq. (1.4.1) has a non-zero right hand side, as in

$$\mathcal{L}[x(t)] \equiv a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t) \,. \tag{1.4.21}$$

Let us introduce the symbol $\mathcal{L}[\phi(t)]$ to indicate linear (differential) operations on any function $\phi(t)$, generalizing simple multiplication by a constant as the simplest linear operation.

The linearity of the equation again leads to a *superposition principle* for solutions of the inhomogeneous equation. Suppose that we have obtained solutions $x_1(t)$ and $x_2(t)$ in the presence of forces $f_1(t)$ and $f_2(t)$, i.e. $\mathcal{L}[x_1(t)] = f_1(t)$ and $\mathcal{L}[x_2(t)] = f_2(t)$. Then the solution for a superposition of forces is obtained simply by superposition of solutions, since

$$\mathcal{L}[c_1 x_1 + c_2 x_2] = c_1 \mathcal{L}[x_1] + c_2 \mathcal{L}[x_2] = c_1 f_1 + c_2 f_2 \,. \tag{1.4.22}$$

Thus if we find a specific class of forcing functions whose solutions are simple, we can find many more solutions by superposition. In fact, such a class of functions are conveniently provided by $\sin(\omega t)$ and $\cos(\omega t)$, as we shall see shortly. Furthermore, according to the *Fourier theorem*, which we shall encounter later on, *any function $f(t)$ can be written as a superposition of sines and cosines.* Thus for a general forcing function $f(t)$, we should first decompose it into a so-called *Fourier series*, and then superpose the solutions in response to the *Fourier components*. The simplest case of forced damped harmonic motion is discussed next.