1.4.5 Steady–state solutions to forced harmonic motion

The generalized equation of motion of a damped harmonic oscillator, subject to an external time dependent force $F(t) = mf(t)$, is

$$\mathcal{L}[x(t)] \equiv \ddot{x} + \gamma\dot{x} + \omega_0^2 = f(t)\,. \tag{1.4.23}$$

We shall look for the solution in response to a force at a single frequency ω. Without loss of generality we can write such a force as $f_\omega \cos(\omega t)$; any mixture of sines and cosines corresponds to a simple shift in t. The complex exponential notation is very useful in this case, and we shall write the starting equation as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \Re\left[f_\omega e^{i\omega t}\right]\,. \tag{1.4.24}$$

It is useful to search for a so-called *steady–state* solution reached after sufficiently long time. Such a solution is likely to have the same period as the external force, and we thus guess (and verify) that it has the form $x_\omega = \Re\left[Ce^{i\omega t}\right]$, where C is a complex number. Substituting this trial solution in the above equation yields

$$\Re\left[\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) Ce^{i\omega t}\right] = \Re\left[f_\omega e^{i\omega t}\right]\,. \tag{1.4.25}$$

The linear operator on the left hand side indeed preserves the complex exponential form of the function, implying that the solution is correct, provided that we choose

$$\begin{aligned} C &= \frac{f_\omega}{\omega_0^2 - \omega^2 + i\gamma\omega} = f_\omega\left[\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} - i\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right] \\ &\equiv Ae^{i\phi} = A\left[\cos\phi + i\sin\phi\right]. \end{aligned} \tag{1.4.26}$$

The steady–state solution can thus be written as $x_\omega = A\cos(\omega t + \phi)$, where the amplitude is

$$A = \frac{f_\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} = \frac{Qf_\omega}{\omega\omega_0}\frac{1}{\sqrt{Q^2\left(\omega_0/\omega - \omega/\omega_0\right)^2 + 1}}\,. \tag{1.4.27}$$

The amplitude goes from the static result f_ω/ω_0^2 at $\omega = 0$, to f_ω/ω^2 as $\omega \to \infty$. In between it has a maximum value of close to Qf_ω/ω_0^2 at $\omega \approx \omega_0$. (The precise location of the maximum is in fact somewhat smaller than ω_0.) The sharpness of the peak is determined by Q, with larger Q giving a narrower and larger maximum. The phase ϕ is a solution to [10]

$$\phi = -\tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = -\tan^{-1}\left(\frac{Q^{-1}}{\omega_0/\omega - \omega/\omega_0}\right)\,. \tag{1.4.28}$$

[10] Note that because of the periodic nature of $\tan\phi$, there is some ambiguity to choice of angle. However, physical reasoning suggests that the oscillations should lag the force, rather than anticipate it. Thus the appropriate phase angle goes from goes from 0 at $\omega = 0$ to $-\pi/2$ at $\omega = \omega_0$, and continues to $-\pi$ as $\omega \to \infty$.