1.4.6 Resonance

The above results indicate that the oscillator responds with the largest amplitude at a frequency close to its natural, or *resonance*, frequency. The highest amplitudes are achieved for high quality systems, although in such cases the forcing frequency has to be carefully tuned since the resonant response occurs in a very narrow frequency range.

The steady–state amplitude can be large, but is finite, as long as $\gamma \neq 0$ <u>and</u> $\omega \neq \omega_0$. For $\gamma = 0$, the general solution presented earlier becomes infinite for $\omega = \omega_0$, and is therefore inapplicable. We go back to the differential equation, which in this limit reads

$$\ddot{x} + \omega_0^2 x = \Re\left[f_{\omega_0} e^{i\omega_0 t}\right] . \tag{1.4.29}$$

Since the simple complex exponential leads to infinities, as in Eq.(1.4.15) we multiply the exponential by a power of t and use as trial solution

$$x(t) = \Re\left[cte^{i\omega_0 t}\right] , \text{ with } \quad \dot{x}(t) = \Re\left[c(1 + i\omega_0 t)e^{i\omega_0 t}\right] , \text{ and } \quad \ddot{x}(t) = \Re\left[c(2i\omega_0 - \omega_0^2 t)e^{i\omega_0 t}\right] . \tag{1.4.30}$$

Substituting into the equation of motion confirms that this is indeed a solution, provided that we choose

$$2i\omega_0 c = f_{\omega_0}, \quad \Rightarrow \quad c = -i\frac{f_{\omega_0}}{2\omega_0} . \tag{1.4.31}$$

The full solution at resonance is thus

$$x_{\omega_0} = \Re\left[-i\frac{f_{\omega_0}}{2\omega_0} t e^{i\omega_0 t}\right] = \frac{f_{\omega_0}}{2\omega_0}\, t \, \cos(\omega_0 t - \pi/2) . \tag{1.4.32}$$

The resonant solution has an amplitude that grows linearly with time. This linear growth will eventually be stopped by damping, or other physical constraints. The phase of this solution lags the force by $\pi/2$, but note that the velocity $\dot{x}(t) \approx f_{\omega_0} t \cos(\omega_0 t)/2$ is in phase with the external force, facilitating the input of energy into the system, as discussed next.