1.4.7 Transients

The steady–state solution is reached after all knowledge of the initial state of the system is lost. To obtain the full solution at all times, we incorporate the initial conditions as follows: First, note that adding any solution of the homogeneous differential equation (i.e. with no forcing) to the steady–state result, gives a new solution for the forced equation. We thus arrive at the following principle for inhomogeneous linear equations: *The general solution for the inhomogeneous linear equation* is *any particular (e.g. steady–state) solution* of the inhomogeneous equation plus *the general solution to the homogeneous equation.*

Going back to

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_\omega \cos(\omega t) = \Re\left[f_\omega e^{i\omega t}\right] , \tag{1.4.33}$$

we can write the most general solution as

$$x(t) = A\cos\left(\omega t + \phi\right) + \tilde{A}e^{-\gamma t/2}\cos\left(\tilde{\omega}t + \tilde{\phi}\right) . \tag{1.4.34}$$

The first term is the steady–state solution oscillating at the forcing frequency of ω, with A and ϕ fixed by the forcing function as in Eqs.(1.4.27) and (1.4.28). The second term is the general solution to the homogeneous equation, which depends on two as yet unknown parameters $\tilde{A}$ and $\tilde{\phi}$. Its frequency is $\tilde{\omega} = \omega_0\sqrt{1 - Q^{-2}/4}$, the natural oscillation frequency of the (free) system. The second term describes *transients* which eventually decay to zero, influencing only the initial behavior.

The two unknown parameters in the transient can be determined by specifying two initial conditions. For simplicity, let us assume that the system starts at rest, with $x(t = 0) = \dot{x}(t = 0) = 0$. Using complex exponential, we have

$$\begin{cases} x(t) = \Re\left[Ce^{i\omega t} + \tilde{C}e^{(i\tilde{\omega} - \gamma/2)t}\right] \\ \dot{x}(t) = \Re\left[i\omega Ce^{i\omega t} + \left(i\tilde{\omega} - \gamma/2\right)\tilde{C}e^{(i\tilde{\omega} - \gamma/2)t}\right] \end{cases} . \tag{1.4.35}$$

The first equation at $t = 0$ yields

$$0 = \Re\left[C + \tilde{C}\right] = C_\Re + \tilde{C}_\Re, \qquad \Rightarrow \qquad \tilde{C}_\Re = -C_\Re = -A\cos\phi\,, \tag{1.4.36}$$

where we have explicitly written the complex coefficients in terms of their real and imaginary parts, as $C = C_\Re + iC_\Im$. The imaginary part of $\tilde{C}$ is obtained from the second initial condition

$$0 = \Re\left[i\omega\left(C_\Re + iC_\Im\right) + \left(i\tilde{\omega} - \frac{\gamma}{2}\right)\left(\tilde{C}_\Re + i\tilde{C}_\Im\right)\right] = -\omega C_\Im - \tilde{\omega}\tilde{C}_\Im - \frac{\gamma}{2}\tilde{C}_\Re\,, \tag{1.4.37}$$

as

$$\tilde{C}_\Im = -\frac{\omega}{\tilde{\omega}}C_\Im + \frac{\gamma}{2\tilde{\omega}}C_\Re = -A\left(\frac{\omega}{\tilde{\omega}}\sin\phi - \frac{\gamma}{2\tilde{\omega}}\cos\phi\right) . \tag{1.4.38}$$

The full solution can thus be written as

$$x(t) = A\cos\left(\omega t + \phi\right) - Ae^{-\gamma t/2}\left[\cos\phi\cos\left(\tilde{\omega}t\right) - \left(\frac{\omega}{\tilde{\omega}}\sin\phi - \frac{\gamma}{2\tilde{\omega}}\cos\phi\right)\sin\left(\tilde{\omega}t\right)\right] . \tag{1.4.39}$$

We can make the following observations on this solution:

- The transients decay to zero after a time $\tau \propto \gamma^{-1}$. During this time, a high quality system may have many oscillations, and the superposition of signals can lead to patterns that appear irregular.

- The initial amplitude of the transient is of the same order as that of the steady state, and in particular $\tilde{A} \to A$ as $\gamma \to 0$. This can lead situations where the solution at first becomes larger than its final value (known as *overshoot*) before decreasing to its steady state value.

- At driving frequencies close to the natural frequency of the system, the initial transients can lead to beats. In particular, for $\gamma \to 0$, we have

$$x(t) = A\left[\cos\left(\omega t\right) - \cos\left(\tilde{\omega} t\right)\right] = 2A\sin\left(\frac{\omega - \tilde{\omega}}{2}t\right)\cos\left(\frac{\omega + \tilde{\omega}}{2}t\right),$$

i.e. a beating frequency of $\omega - \tilde{\omega}$.