## Recap

⋆ Damped oscillations are described by the linear differential equation $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$.

⋆ The character of the solution depends on the *quality factor* $Q = \omega_0/\gamma$.

⋆ Subject to $x(t=0) = 0$ and $\dot{x}(t=0) = v_0$, solutions are:

1. Over-damped motion: $x(t) = v_0 e^{-\gamma t/2}[\sinh{(s\,t)}/s]$, with $s = \sqrt{\gamma^2/4 - \omega_0^2}$
2. Critical damping: $x(t) = v_0 t e^{-\gamma t/2}$.
3. Under-damped motion: $x(t) = v_0 e^{-\gamma t/2}[\sin(\tilde{\omega}t)/\tilde{\omega}]$, with $\tilde{\omega} = \omega_0\sqrt{1 - 1/(4Q^2)}$.

⋆ Harmonically forced, damped linear oscillations satisfy

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_\omega \cos(\omega t) = \Re\left[f_\omega e^{i\omega t}\right], \quad \text{with} \qquad f_\omega = F_\omega/m.$$

⋆ Steady–state solutions to this equation can be written as

$$x(t) = \Re\left[C e^{i\omega t}\right] = A\cos\left(\omega t + \phi\right),$$

with

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + i\gamma\omega} = f_0\left[\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} - i\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right],$$

giving (with $Q = \omega_0/\gamma$)

$$A = \frac{Qf_0}{\omega\omega_0}\frac{1}{\sqrt{Q^2\left(\omega_0/\omega - \omega/\omega_0\right)^2 + 1}}, \quad \text{and} \quad \phi = -\tan^{-1}\left(\frac{Q^{-1}}{\omega_0/\omega - \omega/\omega_0}\right).$$