1.4 General linear ordinary differential equations

1.4.1 General solution

The general form of an mth order, linear, homogeneous[7], differential equation is

$$a_m \frac{d^m x}{dt^m} + a_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0 \,, \tag{1.4.1}$$

where $\{a_m, \cdots, a_0\}$ are fixed parameters. Given our past success with the exponential function, we can guess that a particular solution to this equation should be of the form $x(t) = ce^{\lambda t}$. It is then easy to check that each subsequent derivative multiplies $x(t)$ by a factor λ, such that

$$\frac{d^m x}{dt^m} = \lambda^m x(t) \,. \tag{1.4.2}$$

Substituting this result into the differential equation gives

$$\left[a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0\right] x(t) = 0 \,. \tag{1.4.3}$$

Allowed values of λ are obtained by solving for where the expression in the square brackets is zero. In fact, this mth order algebraic equation has m solutions, which we shall label $\{\lambda_1, \cdots, \lambda_m\}$. Any of these values gives an acceptable particular solution.

An important property of homogeneous linear ODEs is that a general solution obtained by adding the particular solutions. It is valuable to remember that this *superposition principle* applies to linear systems only, and fails if any non-linearity is present. The most general solution of Eq. (1.4.1) is thus given by

$$x(t) = c_1 e^{\lambda_1 t} + \cdots + c_m e^{\lambda_m t} \,. \tag{1.4.4}$$

As anticipated, this solution depends on m independent parameters $\{c_1, \cdots, c_m\}$. In the description of most physical systems, the parameters $\{a_m, \cdots, a_0\}$ describing the ODE, and hence the coefficients in the polynomial equation

$$a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0 = 0 \tag{1.4.5}$$

are real numbers. This implies that if λ_i is a particular solution of this equation, so is its complex conjugate λ_i^*. Thus solutions to the equation are either real, or appear in complex conjugate pairs $a \pm i\omega$. The latter type of solutions can be combined to provide real solutions (such as $e^{at} \sin(\omega t)$) that oscillate with frequency ω, and grow or decay exponentially in time. The damped harmonic motion discussed next provides an important prototype of such behavior.

[7] For inhomogeneous equations, to be discussed later, the right hand side is not zero.

1.4.2 General Damped Harmonic Motion

We noted earlier that if variations in time do not change (conserve) the 'energy' $E = \dot{x}^2/2 + V(x)$, the resulting motion is governed by the second order ODE, $\ddot{x} + V'(x) = 0$, which satisfies time reversal symmetry. Damping is, however, present in mechanical systems, causing irreversible loss of energy. Friction, air drag, viscosity, are all forms of energy dissipation, and the resulting energy loss is in principle a complex function of the motion $x(t)$. We again appeal to instantaneity and continuity to postulate that for continuous motion dE/dt is a function of velocity $\dot{x}$, whose series expansion must start with a lowest order term proportional to $-\dot{x}^2$. (A constant term would cause continuous decrease of energy in the absence of motion, while a linear term can lead to increase of energy depending on the sign of $\dot{x}$.) The loss of energy is then expressed as[8]

$$-\gamma\dot{x}^2 = \frac{dE}{dt} = \frac{d}{dt}\left(\frac{\dot{x}^2}{2} + V(x)\right) = \dot{x}\left[\ddot{x} + V'(x)\right], \quad \Rightarrow \quad \ddot{x} = -V'(x) - \gamma\dot{x}. \tag{1.4.6}$$

The last equation could have been also obtained by equating the net force to the acceleration, assuming a friction force $F_v = b\dot{x}$. Small amplitude deformations, approximated as in Eq. (1.1.6), in the presence of damping, are now described by the linear differential equation

$$m\ddot{x} + b\dot{x} + Kx = 0\,, \tag{1.4.7}$$

which can again be brought to the more standard form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\,, \tag{1.4.8}$$

by setting $\gamma = b/m$, and $\omega_0^2 = K/m$.

Following the general scheme, we can seek particular solutions to the above equation by trying the exponential form $x = ce^{\lambda t}$. Since $\dot{x} = \lambda x$ and $\ddot{x} = \lambda^2 x$, λ must be a solution to the quadratic equation

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0\,. \tag{1.4.9}$$

The two solutions to this equation are

$$\lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,. \tag{1.4.10}$$

The character of the solutions changes dramatically depending on whether the quantity under the square–root is positive or negative. This in turn is controlled by the ratio

$$Q = \frac{\omega_0}{\gamma},$$

which is commonly referred to as the *quality factor*. We shall describe the three classes of possible solutions in turn.

[8]Recall that according to the chain rule in Eq. (1.1.23), $\frac{d}{dt}V(x) = \frac{dV}{dx}\frac{dx}{dt}$.

1.4.3 Forms of Damped Motion

1. Over-damped motion: For $\gamma > 2\omega_0$ ($Q < 1/2$), the solutions for λ are *real*, and can be written as

$$\lambda_\pm = -\frac{\gamma}{2} \pm s, \qquad \text{where} \qquad s = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2}\sqrt{1-4Q^2}\,. \tag{1.4.11}$$

Note that s is a real number, with λ_+ and λ_- both negative. Each choice of λ thus results in an exponentially damped solution, eventually decaying to zero at large times. For this reason, the over-damped behavior is also called *dead beat*. Naturally, the very long time behavior is controlled by the slower of the two exponential decays corresponding to λ_+.

As a specific example, consider a system launched from the equilibrium point $x = 0$ at time $t = 0$, with the initial velocity $\dot{x}(t = 0) = v_0$. The general solution to the motion is obtained by the superposition of the two exponentials, as

$$\begin{cases} x(t) = c_+ e^{-\frac{\gamma}{2}t+st} + c_- e^{-\frac{\gamma}{2}t-st}, \\ \dot{x}(t) = c_+\left(-\frac{\gamma}{2}+s\right)e^{-\frac{\gamma}{2}t+st} + c_-\left(-\frac{\gamma}{2}-s\right)e^{-\frac{\gamma}{2}t-st}. \end{cases} \tag{1.4.12}$$

From $x(0) = 0$, we get $c_- = -c_+$, while

$$\dot{x}(0) = v_0 = c_+\left[-\frac{\gamma}{2} - s + \frac{\gamma}{2} + s\right] = 2c_+s, \quad \Rightarrow \quad c_+ = \frac{v_0}{2s}\,. \tag{1.4.13}$$

The full solution is thus given by

$$x(t) = \frac{v_0}{2s}\left(e^{-\frac{\gamma}{2}t+st} - e^{-\frac{\gamma}{2}t-st}\right) = v_0 e^{-\gamma t/2}\frac{\sinh(st)}{s} = v_0 e^{-\gamma t/2}\frac{\sinh\left(\sqrt{\gamma^2/4-\omega_0^2}\,t\right)}{\sqrt{\gamma^2/4-\omega_0^2}}\,. \tag{1.4.14}$$

2. Critical damping: At the special value of $Q = \omega_0/\gamma = 1/2$ the two solutions for λ merge to the single value of $-\gamma/2$. However, we expect that the second order differential equation should have two independent solutions. We can check that $x_2(t) = te^{-\gamma t/2}$ is also a solution:[9]

$$x_2(t) = e^{-\gamma t/2}t, \Rightarrow \quad \dot{x}_2(t) = e^{-\gamma t/2}\left(-\frac{\gamma}{2}t + 1\right), \Rightarrow \quad \ddot{x}_2(t) = e^{-\gamma t/2}\left(\frac{\gamma^2}{4}t - \frac{\gamma}{2} - \frac{\gamma}{2}\right)\,. \tag{1.4.15}$$

Substituting into the original solution, and using $\omega_0^2 = \gamma^2/4$, confirms that x_2 is indeed a solution.

The general solution in this critically damped case has thus the form

$$x(t) = e^{-\gamma t/2}\left(c_1 + c_2 t\right)\,. \tag{1.4.16}$$

[9]Quite generally when two or more exponents of trial exponential solutions to linear differential equations become equal, we can find new solutions by multiplying the exponentials by powers of t.

For a solution that starts at the origin for $t = 0$, we must have $c_1 = 0$. If the initial velocity is v_0, we thus obtain

$$x(t) = v_0 t e^{-\gamma t/2}\,. \tag{1.4.17}$$

This solution goes through a maximum at time $t_{\rm max} = 2/\gamma$ (from $\dot{x} = 0$), and then decays to the origin. For a fixed γ, as ω_0 is increased from zero, the fastest decay time is obtained at critical damping.

3. Under-damped motion: For $\omega_0 > \gamma/2$ ($Q > 1/2$), the solutions for λ are *complex*, and given by

$$\lambda_\pm = -\frac{\gamma}{2} \pm i\tilde{\omega}, \qquad \text{where} \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}\,. \tag{1.4.18}$$

The general solution to the motion is obtained by the superposition of exponentials, as

$$x(t) = c_+ e^{-\frac{\gamma}{2}t + i\tilde{\omega}t} + c_- e^{-\frac{\gamma}{2}t - i\tilde{\omega}t}.$$

As noted earlier the two exponential solutions are complex conjugates, and a real solution for $x(t)$ can be obtained by setting $c_- = c_+^*$, and written as

$$x(t) = \Re\left[c e^{-\frac{\gamma}{2}t + i\tilde{\omega}t}\right] = \tilde{A} e^{-\gamma t/2}\cos(\tilde{\omega}t + \tilde{\phi})\,, \tag{1.4.19}$$

where in the last step, we have assumed a complex number $c = \tilde{A}e^{i\tilde{\phi}}$, with amplitude $\tilde{A}$, and phase $\tilde{\phi}$.

This solution has the following properties:

- Because of the imaginary exponential, it has an oscillatory character, with a period $T = 2\pi/\tilde{\omega}$. Introduction of damping (finite γ) reduces the frequency from ω_0. The period thus goes up, eventually diverging to infinity as $\gamma \to 2\omega_0$.
- The amplitude of oscillations decays with time as $e^{-\gamma t/2}$. The characteristic decay time is proportional to $1/\gamma$, and is independent of ω_0.
- Over the characteristic decay time $\tau \sim 1/\gamma$, the number of oscillations $N \sim 2\pi/\tilde{\omega}\tau \sim \gamma/\tilde{\omega}$ is roughly set by Q: A high quality oscillator executes many oscillations before dying away, while a low quality one has few oscillations.

The phase of the solution depends on the initial conditions. For example, starting with $x(t = 0) = 0$ and $\dot{x}(t = 0) = v_0$ gives

$$x(t) = v_0 e^{-\gamma t/2}\frac{\sin(\tilde{\omega}t)}{\tilde{\omega}}\,. \tag{1.4.20}$$

(This solution is the *analytic continuation* of that obtained in the over-damped case, with $s \to i\tilde{\omega}$.)

1.4.4 Inhomogeneous linear ODEs

The *inhomogeneous* variant of the nth order, linear equation in Eq. (1.4.1) has a non-zero right hand side, as in

$$\mathcal{L}[x(t)] \equiv a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)\,. \tag{1.4.21}$$

Let us introduce the symbol $\mathcal{L}[\phi(t)]$ to indicate linear (differential) operations on any function $\phi(t)$, generalizing simple multiplication by a constant as the simplest linear operation.

The linearity of the equation again leads to a *superposition principle* for solutions of the inhomogeneous equation. Suppose that we have obtained solutions $x_1(t)$ and $x_2(t)$ in the presence of forces $f_1(t)$ and $f_2(t)$, i.e. $\mathcal{L}[x_1(t)] = f_1(t)$ and $\mathcal{L}[x_2(t)] = f_2(t)$. Then the solution for a superposition of forces is obtained simply by superposition of solutions, since

$$\mathcal{L}[c_1 x_1 + c_2 x_2] = c_1 \mathcal{L}[x_1] + c_2 \mathcal{L}[x_2] = c_1 f_1 + c_2 f_2\,. \tag{1.4.22}$$

Thus if we find a specific class of forcing functions whose solutions are simple, we can find many more solutions by superposition. In fact, such a class of functions are conveniently provided by $\sin(\omega t)$ and $\cos(\omega t)$, as we shall see shortly. Furthermore, according to the *Fourier theorem*, which we shall encounter later on, *any function $f(t)$ can be written as a superposition of sines and cosines.* Thus for a general forcing function $f(t)$, we should first decompose it into a so-called *Fourier series*, and then superpose the solutions in response to the *Fourier components*. The simplest case of forced damped harmonic motion is discussed next.

1.4.5 Steady–state solutions to forced harmonic motion

The generalized equation of motion of a damped harmonic oscillator, subject to an external time dependent force $F(t) = mf(t)$, is

$$\mathcal{L}[x(t)] \equiv \ddot{x} + \gamma \dot{x} + \omega_0^2 = f(t)\,. \tag{1.4.23}$$

We shall look for the solution in response to a force at a single frequency ω. Without loss of generality we can write such a force as $f_\omega \cos(\omega t)$; any mixture of sines and cosines corresponds to a simple shift in t. The complex exponential notation is very useful in this case, and we shall write the starting equation as

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \Re\left[f_\omega e^{i\omega t}\right]\,. \tag{1.4.24}$$

It is useful to search for a so-called *steady–state* solution reached after sufficiently long time. Such a solution is likely to have the same period as the external force, and we thus guess (and verify) that it has the form $x_\omega = \Re\left[Ce^{i\omega t}\right]$, where C is a complex number. Substituting this trial solution in the above equation yields

$$\Re\left[\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) Ce^{i\omega t}\right] = \Re\left[f_\omega e^{i\omega t}\right]\,. \tag{1.4.25}$$

The linear operator on the left hand side indeed preserves the complex exponential form of the function, implying that the solution is correct, provided that we choose

$$\begin{aligned} C &= \frac{f_\omega}{\omega_0^2 - \omega^2 + i\gamma\omega} = f_\omega \left[\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} - i\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} \right] \\ &\equiv Ae^{i\phi} = A\left[\cos\phi + i\sin\phi\right]. \end{aligned} \tag{1.4.26}$$

The steady–state solution can thus be written as $x_\omega = A\cos(\omega t + \phi)$, where the amplitude is

$$A = \frac{f_\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} = \frac{Qf_\omega}{\omega\omega_0}\frac{1}{\sqrt{Q^2\left(\omega_0/\omega - \omega/\omega_0\right)^2 + 1}}. \tag{1.4.27}$$

The amplitude goes from the static result f_ω/ω_0^2 at $\omega = 0$, to f_ω/ω^2 as $\omega \to \infty$. In between it has a maximum value of close to Qf_ω/ω_0^2 at $\omega \approx \omega_0$. (The precise location of the maximum is in fact somewhat smaller than ω_0.) The sharpness of the peak is determined by Q, with larger Q giving a narrower and larger maximum. The phase ϕ is a solution to [10]

$$\phi = -\tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = -\tan^{-1}\left(\frac{Q^{-1}}{\omega_0/\omega - \omega/\omega_0}\right). \tag{1.4.28}$$

1.4.6 Resonance

The above results indicate that the oscillator responds with the largest amplitude at a frequency close to its natural, or *resonance*, frequency. The highest amplitudes are achieved for high quality systems, although in such cases the forcing frequency has to be carefully tuned since the resonant response occurs in a very narrow frequency range.

The steady–state amplitude can be large, but is finite, as long as $\gamma \neq 0$ and $\omega \neq \omega_0$. For $\gamma = 0$, the general solution presented earlier becomes infinite for $\omega = \omega_0$, and is therefore inapplicable. We go back to the differential equation, which in this limit reads

$$\ddot{x} + \omega_0^2 x = \Re\left[f_{\omega_0} e^{i\omega_0 t}\right]. \tag{1.4.29}$$

Since the simple complex exponential leads to infinities, as in Eq.(1.4.15) we multiply the exponential by a power of t and use as trial solution

$$x(t) = \Re\left[cte^{i\omega_0 t}\right], \text{ with } \quad \dot{x}(t) = \Re\left[c(1 + i\omega_0 t)e^{i\omega_0 t}\right], \text{ and } \quad \ddot{x}(t) = \Re\left[c(2i\omega_0 - \omega_0^2 t)e^{i\omega_0 t}\right]. \tag{1.4.30}$$

Substituting into the equation of motion confirms that this is indeed a solution, provided that we choose

$$2i\omega_0 c = f_{\omega_0}, \quad \Rightarrow \quad c = -i\frac{f_{\omega_0}}{2\omega_0}. \tag{1.4.31}$$

[10] Note that because of the periodic nature of $\tan\phi$, there is some ambiguity to choice of angle. However, physical reasoning suggests that the oscillations should lag the force, rather than anticipate it. Thus the appropriate phase angle goes from goes from 0 at $\omega = 0$ to $-\pi/2$ at $\omega = \omega_0$, and continues to $-\pi$ as $\omega \to \infty$.

The full solution at resonance is thus

$$x_{\omega_0} = \Re\left[-i\frac{f_{\omega_0}}{2\omega_0} t e^{i\omega_0 t}\right] = \frac{f_{\omega_0}}{2\omega_0}\, t\, \cos(\omega_0 t - \pi/2)\,. \tag{1.4.32}$$

The resonant solution has an amplitude that grows linearly with time. This linear growth will eventually be stopped by damping, or other physical constraints. The phase of this solution lags the force by $\pi/2$, but note that the velocity $\dot{x}(t) \approx f_{\omega_0} t \cos(\omega_0 t)/2$ is in phase with the external force, facilitating the input of energy into the system, as discussed next.

1.4.7 Transients

The steady–state solution is reached after all knowledge of the initial state of the system is lost. To obtain the full solution at all times, we incorporate the initial conditions as follows: First, note that adding any solution of the homogeneous differential equation (i.e. with no forcing) to the steady–state result, gives a new solution for the forced equation. We thus arrive at the following principle for inhomogeneous linear equations: *The general solution for the inhomogeneous linear equation* is *any particular (e.g. steady–state) solution* of the inhomogeneous equation plus *the general solution to the homogeneous equation.*

Going back to

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_\omega \cos(\omega t) = \Re\left[f_\omega e^{i\omega t}\right]\,, \tag{1.4.33}$$

we can write the most general solution as

$$x(t) = A\cos(\omega t + \phi) + \tilde{A}e^{-\gamma t/2}\cos\left(\tilde{\omega}t + \tilde{\phi}\right)\,. \tag{1.4.34}$$

The first term is the steady–state solution oscillating at the forcing frequency of ω, with A and ϕ fixed by the forcing function as in Eqs.(1.4.27) and (1.4.28). The second term is the general solution to the homogeneous equation, which depends on two as yet unknown parameters $\tilde{A}$ and $\tilde{\phi}$. Its frequency is $\tilde{\omega} = \omega_0\sqrt{1 - Q^{-2}/4}$, the natural oscillation frequency of the (free) system. The second term describes *transients* which eventually decay to zero, influencing only the initial behavior.

The two unknown parameters in the transient can be determined by specifying two initial conditions. For simplicity, let us assume that the system starts at rest, with $x(t=0) = \dot{x}(t=0) = 0$. Using complex exponential, we have

$$\begin{cases} x(t) = \Re\left[Ce^{i\omega t} + \tilde{C}e^{(i\tilde{\omega}-\gamma/2)t}\right] \\ \dot{x}(t) = \Re\left[i\omega Ce^{i\omega t} + (i\tilde{\omega} - \gamma/2)\,\tilde{C}e^{(i\tilde{\omega}-\gamma/2)t}\right] \end{cases}. \tag{1.4.35}$$

The first equation at $t = 0$ yields

$$0 = \Re\left[C + \tilde{C}\right] = C_\Re + \tilde{C}_\Re, \qquad \Rightarrow \qquad \tilde{C}_\Re = -C_\Re = -A\cos\phi\,, \tag{1.4.36}$$

where we have explicitly written the complex coefficients in terms of their real and imaginary parts, as $C = C_{\Re}+iC_{\Im}$. The imaginary part of $\tilde{C}$ is obtained from the second initial condition

$$0 = \Re\left[i\omega\left(C_{\Re}+iC_{\Im}\right)+\left(i\tilde{\omega}-\frac{\gamma}{2}\right)\left(\tilde{C}_{\Re}+i\tilde{C}_{\Im}\right)\right] = -\omega C_{\Im}-\tilde{\omega}\tilde{C}_{\Im}-\frac{\gamma}{2}\tilde{C}_{\Re}\,, \tag{1.4.37}$$

as

$$\tilde{C}_{\Im} = -\frac{\omega}{\tilde{\omega}}C_{\Im}+\frac{\gamma}{2\tilde{\omega}}C_{\Re} = -A\left(\frac{\omega}{\tilde{\omega}}\sin\phi-\frac{\gamma}{2\tilde{\omega}}\cos\phi\right)\,. \tag{1.4.38}$$

The full solution can thus be written as

$$x(t) = A\cos\left(\omega t+\phi\right)-Ae^{-\gamma t/2}\left[\cos\phi\cos\left(\tilde{\omega}t\right)-\left(\frac{\omega}{\tilde{\omega}}\sin\phi-\frac{\gamma}{2\tilde{\omega}}\cos\phi\right)\sin\left(\tilde{\omega}t\right)\right]\,. \tag{1.4.39}$$

We can make the following observations on this solution:

- The transients decay to zero after a time $\tau \propto \gamma^{-1}$. During this time, a high quality system may have many oscillations, and the superposition of signals can lead to patterns that appear irregular.

- The initial amplitude of the transient is of the same order as that of the steady state, and in particular $\tilde{A} \to A$ as $\gamma \to 0$. This can lead situations where the solution at first becomes larger than its final value (known as *overshoot*) before decreasing to its steady state value.

- At driving frequencies close to the natural frequency of the system, the initial transients can lead to beats. In particular, for $\gamma \to 0$, we have

  $$x(t) = A\left[\cos\left(\omega t\right)-\cos\left(\tilde{\omega}t\right)\right] = 2A\sin\left(\frac{\omega-\tilde{\omega}}{2}t\right)\cos\left(\frac{\omega+\tilde{\omega}}{2}t\right),$$

  i.e. a beating frequency of $\omega-\tilde{\omega}$.

Recap

⋆ Damped oscillations are described by the linear differential equation $\ddot{x}+\gamma\dot{x}+\omega_0^2x = 0$.
⋆ The character of the solution depends on the *quality factor* $Q = \omega_0/\gamma$.
⋆ Subject to $x(t=0) = 0$ and $\dot{x}(t=0) = v_0$, solutions are:

1. Over-damped motion: $x(t) = v_0e^{-\gamma t/2}[\sinh\left(s\,t\right)/s]$, with $s = \sqrt{\gamma^2/4-\omega_0^2}$
2. Critical damping: $x(t) = v_0te^{-\gamma t/2}$.
3. Under-damped motion: $x(t) = v_0e^{-\gamma t/2}[\sin(\tilde{\omega}t)/\tilde{\omega}]$, with $\tilde{\omega} = \omega_0\sqrt{1-1/(4Q^2)}$.

⋆ Harmonically forced, damped linear oscillations satisfy

$$\ddot{x}+\gamma\dot{x}+\omega_0^2x = f_\omega\cos(\omega t) = \Re\left[f_\omega e^{i\omega t}\right], \quad \text{with} \qquad f_\omega = F_\omega/m.$$

⋆ Steady–state solutions to this equation can be written as

$$x(t) = \Re\left[Ce^{i\omega t}\right] = A\cos\left(\omega t + \phi\right),$$

with

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + i\gamma\omega} = f_0\left[\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} - i\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right],$$

giving (with $Q = \omega_0/\gamma$)

$$A = \frac{Qf_0}{\omega\omega_0}\frac{1}{\sqrt{Q^2\left(\omega_0/\omega - \omega/\omega_0\right)^2 + 1}}, \quad \text{and} \quad \phi = -\tan^{-1}\left(\frac{Q^{-1}}{\omega_0/\omega - \omega/\omega_0}\right).$$