Chapter 2

Multiple variables

2.1 Two variables

2.1.1 First order coupled ODEs

While the position of a particle along a line can be represented by a single coordinate, its location on a two-dimensional plane requires two coordinates, say indicated by x_1 and x_2. In the absence of time-reversal symmetry, the generalization of Eq. (1.1.10) to two degrees of freedom is (setting $\mu = 1$ without loss of generality)

$$\dot{x}_1 = F_1(x_1, x_2), \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2)\,. \tag{2.1.1}$$

Let us assume that $x_1 = x_2 = 0$ is a point of equilibrium (at which $F_1 = F_2 = 0$). Series expansions[1] of the force around this point then yield to the lowest order

$$F_1(x_1, x_2) = f_{11}x_1 + f_{12}x_2 + \cdots\,, \qquad \text{and} \qquad F_2(x_1, x_2) = f_{21}x_1 + f_{22}x_2 + \cdots\,. \tag{2.1.4}$$

Understanding the behavior of the system near $x_1 = x_2 = 0$ thus requires solving the pair of coupled first order ODEs

$$\begin{cases} \dot{x}_1 = f_{11}x_1 + f_{12}x_2 \\ \dot{x}_2 = f_{21}x_1 + f_{22}x_2 \end{cases}, \quad \Longrightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \Longrightarrow \quad \frac{d\vec{x}}{dt} = \mathbf{F}\cdot\vec{x}\,. \tag{2.1.5}$$

[1] The Taylor expansion of a function of two variables takes the form

$$\phi(x,y) = \phi_{00} + \phi_{10}x + \phi_{01}y + \frac{\phi_{20}}{2}x^2 + \phi_{11}xy + \frac{\phi_{02}}{2}y^2 + \cdots \equiv \sum_{m,n} \phi_{mn}\frac{x^m}{m!}\frac{y^n}{n!}\,, \tag{2.1.2}$$

with the coefficients obtained from mixed derivatives, as

$$\phi_{mn} = \frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\phi(x,y)\Big|_{x=y=0}\,. \tag{2.1.3}$$

Note that the symbol $\partial/\partial x$ is used in place of d/dx, indicating partial derivatives of the function with respect to the variable x, when other variables of the function are held constant.

Note that the linear set of equations can be cast in the form of a 2×2 matrix acting on the column vector composed from x_1 and x_2.

To gain insight, let us first consider a particle sliding down a two-dimensional potential shaped like an ellipsoidal bowl. If we align the coordinates x_1 and x_2 to the axes of the ellipse, the expansion of the potential around its minimum at $x_1 = x_2 = 0$ reads

$$V(x_1, x_2) = k_1 \frac{x_1^2}{2} + k_2 \frac{x_2^2}{2} + \cdots , \tag{2.1.6}$$

where k_x and k_y are the inverse radii of curvature of the bowl. A particle that starts along one of the axes of ellipse, say from $(x_1 = x_1^0, x_2 = 0)$, will stay on the axis as the symmetry $x_2 \to -x_2$ does not select one direction over the other ($F_2 = -k_2 x_2 = 0$ for $x_2 = 0$), and proceeds according to

$$\dot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -k_1 x_1 , \quad \Longrightarrow \quad x_1(t) = x_1^0 e^{-k_1 t} . \tag{2.1.7}$$

A corresponding solution can be obtained for a particle that starts at $(x_1 = 0, x_2 = x_2^0)$ at $t = 0$. The general solution, starting from any initial point is the simple superposition of the two, decaying to zero as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1^0 e^{-k_1 t} \\ x_2^0 e^{-k_2 t} \end{pmatrix} . \tag{2.1.8}$$

Let us suppose, however, that by some curious oversight we had chosen to align our coordinate system at 45° (or some other angle) with respect to the natural directions of the elliptical bowl. The new coordinates $\{x'_1, x'_2\}$, and the old ones are related by[2]

$$\begin{cases} x'_1 = \dfrac{x_1 + x_2}{\sqrt{2}} \\ x'_2 = \dfrac{-x_1 + x_2}{\sqrt{2}} \end{cases} , \qquad \text{and} \qquad \begin{cases} x_1 = \dfrac{x'_1 - x'_2}{\sqrt{2}} \\ x_2 = \dfrac{x'_1 + x'_2}{\sqrt{2}} \end{cases} . \tag{2.1.10}$$

In terms of the new coordinates, the potential energy is

$$V = \frac{k_1}{4} \left(x'_1 - x'_2\right)^2 + \frac{k_2}{4} \left(x'_1 + x'_2\right)^2 = \frac{k_1 + k_2}{4} \left(x_1'^2 + x_2'^2\right) - \left(\frac{k_1 - k_2}{2}\right) x'_1 x'_2 . \tag{2.1.11}$$

The corresponding equations of motion for gradient descent,

$$\begin{cases} \dot{x}'_1 = -\dfrac{\partial V}{\partial x'_1} = -\dfrac{k_1 + k_2}{2} x'_1 + \dfrac{k_1 - k_2}{2} x'_2 \\ \dot{x}'_2 = -\dfrac{\partial V}{\partial x'_2} = -\dfrac{k_1 + k_2}{2} x'_2 + \dfrac{k_1 - k_2}{2} x'_1 \end{cases} , \tag{2.1.12}$$

[2]For a rotation by an angle θ, we have

$$\begin{cases} x' = x \cos\theta + y \sin\theta \\ y' = -x \sin\theta + y \cos\theta \end{cases} , \qquad \text{and} \qquad \begin{cases} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{cases} . \tag{2.1.9}$$

are *coupled* to each other, and a may appear harder to solve. Naturally, from the original solution to the problem, it is easy to construct solutions to these equations as

$$\begin{cases} x_1'(t) = \dfrac{x_1(t) + x_2(t)}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \left[x_1^0 e^{-k_1 t} + x_2^0 e^{-k_2 t} \right] \\ x_2'(t) = \dfrac{x_2(t) - x_1(t)}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \left[x_2^0 e^{-k_2 t} - x_1^0 e^{-k_1 t} \right] \end{cases}. \tag{2.1.13}$$

However, the solutions in the case are not single exponentials, but superposition of two exponentials.

This example demonstrates that there could be a *'right way'* of looking at a system, and many possible *'wrong ways'* (rotated coordinates) of viewing it. The analysis and description of the system becomes much simpler if the right set of coordinates are used. Surprisingly, this is always possible for gradient descent in a linear system, and there is a way to find the right set of variables to describe the problem.