2.1.2 Gradient descent in two dimensions

The most general form of a quadratic potential in two dimensions, generalizing Eq. (2.1.6), is[3]

$$V(x_1, x_2) = V_0 + k_1 \frac{x_1^2}{2} + k_2 \frac{x_2^2}{2} + k_\times x_1 x_2 \,. \tag{2.1.14}$$

Gradient descent in such a potential leads to

$$\begin{cases} \dot{x}_1 = -\dfrac{\partial V}{\partial x_1} = -k_1 x_1 - k_\times x_2 \\ \dot{x}_2 = -\dfrac{\partial V}{\partial x_2} = -k_2 x_2 - k_\times x_1 \end{cases}, \quad \Longrightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 & k_\times \\ k_\times & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} . \tag{2.1.15}$$

The analog of the direction in Eq. (2.1.7) along which the solution proceeds exponentially, is an eigenvector of the above matrix,

$$\mathbf{M} = -\begin{pmatrix} k_1 & k_\times \\ k_\times & k_2 \end{pmatrix} , \tag{2.1.16}$$

with the decay rate provided by the corresponding eigenvalue. In other words, we seek column vectors

$$\vec{e}_\pm \equiv \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \text{such that} \quad \mathbf{M}\vec{e}_\pm = \lambda_\pm \vec{e}_\pm \,. \tag{2.1.17}$$

The indices $\pm$ are in anticipation of there being two directions and corresponding eigenvalues.

To obtain the eigenvalues, the equation is first rearranged as $(\mathbf{M} - \lambda\mathbf{1}) \cdot \vec{e} = \mathbf{0}$, where $\mathbf{1}$ is the unit matrix with ones along the diagonal and zeros elsewhere. For this homogenous set of equations to have a non-zero answer, the *determinant* of the matrix of coefficients has to be zero, i.e.

$$\det(\mathbf{M} - \lambda\mathbf{1}) = 0. \,. \tag{2.1.18}$$

For our 2×2 matrix, this leads to a so-called *characteristic equation* that has the form

$$\lambda^2 - \mathrm{tr}\mathbf{M}\ \lambda + \det\mathbf{M} = 0 \,, \quad \text{with} \quad \mathrm{tr}\mathbf{M} = k_1 + k_2 \quad \text{and} \quad \det\mathbf{M} = k_1 k_2 - k_\times^2 \,. \tag{2.1.19}$$

It is good to recall that the sum of the two eigenvalues is equal to the trace of the matrix, while their product is the determinant of the matrix. Solving this quadratic equation gives

$$\lambda_\pm = -\frac{1}{2}\left[(k_1 + k_2) \pm \sqrt{(k_1 - k_2)^2 + 4k_\times^2}\right] . \tag{2.1.20}$$

Note that the quantity under the square root is strictly positive, indicating that both eigenvalues are real. For stable equilibrium, both eigenvalues should be negative, as positive eigenvalues will take the dynamics to infinity; this occurs for $k_\times^2 > k_1 k_2$ where $\det\mathbf{M} < 0$.

[3]Linear terms in x_1 and x_2 are absent, either because of an inversion symmetry $\vec{x} \to -\vec{x}$, or because we are interested in deviations from a stable equilibrium.