2.1.3 Beyond gradient descent

The reality of the eigenvalues in Eq. (2.1.20) is a consequence of the symmetry of the matrix, which is an inevitable consequence of gradient descent in a quadratic potential. However, even for more complicated potentials gradient descent (that $F_1 = \frac{\partial V}{\partial x_1}$ and $F_2 = \frac{\partial V}{\partial x_2}$) imposes the constraint

$$\frac{\partial F_1}{\partial x_2} = -\frac{\partial^2 V}{\partial x_2 \partial x_1} = -\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial F_2}{\partial x_1}\,, \tag{2.1.21}$$

since mixed partial derivatives can be taken in any order. If so, we may ask what is the outcome of the more general dynamics that does not satisfy the above constraint, e.g. for linearized equations such as in Eq. (2.1.5), where the matrix

$$\mathbf{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\,, \tag{2.1.22}$$

is not symmetric, $f_{12} \neq f_{21}$?

As example, let us consider the following set of equations

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\gamma v - \omega_0^2 x \end{cases}, \quad \Longrightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}\,. \tag{2.1.23}$$

Clearly this systems of two coupled first order equations is simply the damped harmonic oscillator of Eq. (1.4.8) in disguise. The eigenvalues of the asymmetric matrix are given by Eq. (1.4.10). Notably, for $\gamma < 2\omega_0$ the eigenvalues have an imaginary part indicating oscillatory behavior. In the limit $\gamma = 0$, the motion is undamped oscillation (time reversible) and conserves the energy function $E(x, v) = (v^2 + \omega_0^2 x^2)/2$.