### 2.1.4 Hamiltonian evolution

Conservation of energy is an important principle in physics, and it is useful to find a procedure to construct first order equations that conserve some function, say $H\left(x(t), p(t)\right)$. Setting $dH/dt = 0$, and using the chain rule, we need

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} = 0\,. \tag{2.1.24}$$

One way to ensure this condition is to set

$$\dot{x} = \frac{\partial H}{\partial p}\,, \qquad \text{and} \qquad \dot{p} = -\frac{\partial H}{\partial x}\,. \tag{2.1.25}$$

Clearly, Eqs. (2.1.23) for $\gamma = 0$ follow this structure with $v$ playing the role of $p$. Indeed, the *Hamiltonian formulation* of classical equations of motion follow the structure of Eq. (2.1.24) and (2.1.25), with $H(x, p)$ as the total energy in terms of the coordinate $x$ and its conjugate momentum $p$.

Indeed the most general pair of linear ODEs from Eq. (2.1.26) can be recast as

$$\dot{x}_1 = F_1(x_1, x_2) = -\frac{\partial V}{\partial x_1} + \frac{\partial H}{\partial x_2}, \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2) = -\frac{\partial V}{\partial x_2} - \frac{\partial H}{\partial x_1}\,, \tag{2.1.26}$$

as superposition of gradient descent in the potential $V(x_1, x_2)$ with sliding along contours of constant $H(x_1, x_2)$.[4]

[4]For future reference, note that given $F_1$ and $F_2$, the potentials $V$ and $H$ are solutions to

$$\begin{cases} \dfrac{\partial^2 V}{\partial {x_1}^2} + \dfrac{\partial^2 V}{\partial {x_2}^2} = \dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_2}{\partial x_2} & \Rightarrow \quad \nabla^2 V = \nabla \cdot \vec{F} \\ \dfrac{\partial^2 H}{\partial {x_1}^2} + \dfrac{\partial^2 H}{\partial {x_2}^2} = \dfrac{\partial F_1}{\partial x_2} - \dfrac{\partial F_2}{\partial x_1} & \Rightarrow \quad \nabla^2 H = \nabla \times \vec{F} \end{cases}. \tag{2.1.27}$$