2.2 Multiple variables

2.2.1 Many coupled ODEs

The results of the previous section can be generalized to multiple variables indexed by $i = 1, 2, \cdots, n$. The set of coordinates $\{x_i\}$ can be regraded as a point in n dimensional space, and can also be represented as a vector $\vec{x}$ extending from the origin to this point. The generalized equations of motion can now be represented as

$$\dot{x}_i = F_i(\{x_i\}) \text{ for } i = 1, 2, \cdots n\,, \qquad \text{or equivalently as} \quad \dot{\vec{x}} = \vec{F}(\vec{x})\,. \tag{2.2.1}$$

The linearized equations take the form

$$\dot{x}_i = \sum_{j=1}^{n} F_{ij} x_j \text{ for } i = 1, 2, \cdots n\,, \qquad \text{or equivalently as} \quad \dot{\vec{x}} = \mathbf{F}\vec{x}\,, \tag{2.2.2}$$

in terms of the $n \times n$ matrix formed from n^2 elements $\{F_{ij}\}$.

A particular class of linear equations is obtained from gradient descent in a quadratic potential, which can be written as

$$V(\{x_i\}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} x_i x_j\,. \tag{2.2.3}$$

It may appear that n^2 elements are needed to specify the potential. This is in fact not the case since after summation over both i and j, only the symmetric part $(M_{ij} + M_{ji})$ contributes as the coefficient of the term $x_i x_j$, while the antisymmetric part $(M_{ij} - M_{ji})$ vanishes. Thus a general quadratic potential can be represented by $n(n+1)/2$ elements forming a symmetric matrix, in which case $F_{ij} = M_{ij}$ in Eq. (2.2.2).