2.2.2 Indexology

The summation convention (introduced to physics by Einstein) is a convenient way to represent sums, such as appearing in Eq. (2.2.3), in compact form. Basically, any index that appears twice has to be summed over all its possible values.[5] With this convention in mind, Eqs. (2.2.2) and (2.2.3) can be written as

$$\dot{x}_i = F_{ij} x_j \quad \text{and} \quad V = \frac{1}{2} M_{ij} x_i x_j \,, \tag{2.2.4}$$

with now implicit sums over i and j. In applying these rules, it is important to keep the following in mind:

- It is very important to ensure that *any index that represents a component along some direction* appears only 0, 1, 2 times on one side of an equation.
- Any such index that appears once on one side of an equation, must also appear once on the other side of the equation.
- Note that there could be other labels, not indexing components of a vector, that are not subject to the summation rule. The labels for eigenvalues that we shall use shortly are an example of this exemption.

Having introduced the index notation, it is useful to be familiar with the following terminology:

- *Scalars* are quantities that do not carry an index, such as the potential V. They can be constructed by contracting (pairing) of entities with indices, such as in $x_i y_i \equiv \vec{x} \cdot \vec{y}$ (the dot product of vectors $\vec{x}$ and $\vec{y}$), or $M_{ii} \equiv \mathrm{tr}\mathbf{M}$ (trace of a matrix).
- *Vectors* carry a single index such as x_i or $\dot{x}_i$.
- *Matrices* such as M_{ij} can for example be constructed from two vectors, as in $x_i x_j$ or $\dot{x}_i x_j$, or also from product of other matrices, as in $A_{ij} B_{jk} = (AB)_{ik} = C_{ik}$ (the component form of the matrix product $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$).
- *The Kronecker delta-function* δ_{ij} represents the components of the unit matrix, equal to 1 if $i = j$ (along the diagonal) and 0 otherwise (off diagonal). Summing over one index of the delta-function has the effect of replacing it with the other index, as in $\delta_{ij} x_j = x_i$ or $\delta_{ij} M_{jk} = M_{ik}$. Also, note that $\delta_{ii} = n$, where n is the dimensionality of the system.
- We can also construct objects with more indices, such as $x_i x_j x_k$ or $M_{ij} M_{kl}$, sometimes referred to as *tensors* of higher rank (3 and 4 in the two examples).

[5] The more sophisticated Einstein notation, relevant to general relativity, distinguishes between indices appearing as superscripts (upper) or subscripts (lower) on a variable. We shall not deal with this subtlety here.

- We already encountered the *gradient operator* in the contexts of descent in a scalar potential $V(\{x_i\})$. The operation of taking gradient can be represented by the components of the derivative vector $\nabla_i \equiv \frac{\partial}{\partial x_i} \equiv \partial_i$.

- *Vector fields*, such as the previously encountered force $F_i(\{x_i\})$ are vectors whose magnitude and direction vary in coordinate space. The *divergence* of a vector field is the scalar quantity

$$\mathrm{div}\vec{F} = \nabla \cdot \vec{F} = \partial_i F_i \,. \tag{2.2.5}$$

  Note that for $F_i = -\partial_i V$, we find

$$\mathrm{div}\vec{F} = \partial_i F_i = -\partial_i \partial_i V \equiv -\nabla^2 V \,, \tag{2.2.6}$$

  involving the *Laplacian operator* $\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2} = \partial_i \partial_i$.

- The chain rule can also be compactly expressed in this notation as

$$\frac{dV(\{x_i\})}{dt} = \partial_i V \frac{dx_i}{dt} = \dot{x}_i \partial_i V \,, \quad \text{and} \quad \frac{dF_j(\{x_i\})}{dt} = \partial_i F_j \frac{dx_i}{dt} = \dot{x}_i \partial_i F_j \,, \tag{2.2.7}$$

  acting on a scalar and vector respectively.