2.2.3 Eigenvectors and eigenvalues

As noted before, directions along which the solution proceeds as a single exponential, as in Eq. (2.1.7) correspond to eigenvectors of the matrix $\mathbf{F}$ in Eq. (2.2.2). For an $n \times n$ matrix, there are n such eigenvectors that we shall label as $\vec{e}^\alpha$ for $\alpha = 1, 2, \cdots, n$, such that

$$F_{ij} e_j^\alpha = \lambda_\alpha e_i^\alpha \,, \quad \text{for} \quad \alpha = 1, 2, \cdots, n \,, \tag{2.2.8}$$

with e_i^α indicating the components of $\vec{e}^\alpha$, and λ_α as the corresponding eigenvalue. (Note that the index α on the right hand side of the above equation appears twice, but is not summed over, as $\{\lambda_\alpha\}$ do not represent components of a vector, but instead label the solutions of Eq. (2.2.8)).

We noted earlier that eigenvalues of a symmetric matrix with real entries $M_{ij} = M_{ji}$ are real numbers. Let us prove this as an exercise in the summation convention. Multiply both sides of Eq. (2.2.8) with $(e_i\beta)^*$ and sum over i to get

$$(e_i^\beta)^* M_{ij} e_j^\alpha = \lambda_\alpha (e_i^\beta)^* e_i^\alpha \,. \tag{2.2.9}$$

Taking complex conjugates of the above equation, and taking advantage of $M_{ij}^* = M_{ji}$ allows us to rearrange the equation as

$$(e_j^\alpha)^* M_{ji} e_i^\beta = \lambda_\alpha^* e_i^\beta (e_i^\alpha)^* \,. \tag{2.2.10}$$

Noting $M_{ji} e_i^\beta = \lambda_\beta e_j^\beta$, the above equation can be recast as

$$(\lambda_\beta - \lambda_\alpha^*)\, e_i^\beta (e_i^\alpha)^* = 0 \,. \tag{2.2.11}$$

For $\beta = \alpha$, the second term $\sum_i |e_i^\alpha|^2$, the squared magnitude of a (possibly complex) eigenvector $\vec{e}^\alpha$ is explicitly positive. We must therefore have $\lambda_\alpha = \lambda_\alpha^*$, requiring real eigenvalues. In fact both the real and imaginary parts of the vector $\vec{e}^\alpha$ are eigenvectors, and without loss of generality we can limit discussion to real eigenvectors and drop the complex conjugate sign.

For $\alpha \neq \beta$ (and assuming non-degenerate eigenvalues $\lambda_\alpha \neq \lambda_\beta$), we are then lead to another important result, that $\vec{e}^\beta \cdot \vec{e}^\alpha = 0$. The eigenvectors of a real symmetric matrix thus form an orthogonal set in the n-dimensional space. The magnitude of the eigenvectors is arbitrary, but it is useful to make them all equal to unity, such that they form an orthonormal set with $\vec{e}^\beta \cdot \vec{e}^\alpha = \delta_{\alpha\beta}$.

To solve the set of linear ODEs $\dot{x}_i(t) = M_{ij} x_j(t)$, with the initial condition $x_i(t = 0) = x_i(0) = x_i^0$:

- Find the eigenvectors $\vec{e}^\alpha$ and the corresponding eigenvalues λ_α.
- Compute the coordinates of the starting point in the basis formed by the eigenvectors, i.e. $a_\alpha(0) = x_i(0) e_i^\alpha$.
- Each component in the eigenvector basis will evolve as a simple exponential with the corresponding eigenvalue, i.e. $a_\alpha(t) = a_\alpha(0) e^{\lambda_\alpha t}$.

- In terms of these components the location at time t is given by

$$x_i(t) = \sum_\alpha a_\alpha(t) e_i^\alpha = \sum_\alpha a_\alpha(0) e^{\lambda_\alpha t} e_i^\alpha = x_j(0) \sum_\alpha e_j^\alpha e^{\lambda_\alpha t} e_i^\alpha \equiv U_{ij}(t) x_j(0)\,, \qquad (2.2.12)$$

  where we have introduced the linear operator $U_{ij}(t) = \sum_\alpha e_i^\alpha e^{\lambda_\alpha t} e_j^\alpha$ whose action (multiplication) on the initial vector leads to the position at time t.

- For displacements around a stable equilibrium point, the solution in Eq. (2.2.12) must not diverge for any choice of initial condition. For this to hold, all eigenvalues of the matrix must be negative.[6] If the matrix is obtained from gradient descent in the potential $V = K_{ij} x_i x_j / 2$, stability requires all eigenvalues of the matrix to be positive. (The change of sign is due to the negative sign from gradient descent, $F_i = -\partial_i V$.) Such a matrix is called *positive definite* and $K_{ij} x_i x_j > 0$ for any displacement $\vec{x}$.

[6] For non-symmetric matrices with complex eigenvalues, the real parts of all eigenvalues must be negative.