2.3 Higher order coupled linear ODEs

2.3.1 General Form

The generalization of the mth order, linear, homogeneous ODE in Eq. (1.4.1) from a scalar $x(t)$ to an n-component vector $\vec{x}$ takes the form

$$\mathbf{a}_m \frac{d^m \vec{x}}{dt^m} + \mathbf{a}_{m-1} \frac{d^{m-1}\vec{x}}{dt^{m-1}} + \cdots + \mathbf{a}_1 \frac{d\vec{x}}{dt} + \mathbf{a}_0 \vec{x} = 0 \,, \tag{2.3.1}$$

where $\{\mathbf{a}_m, \cdots, \mathbf{a}_0\}$ are now $n \times n$ matrices. Note that the first order ODE set of last section, e.g. in Eq. (2.2.2), correspond to the choice of $\mathbf{a}_0 = \mathbf{M}$, $\mathbf{a}_1 = \mathbf{1}$, and $\mathbf{a}_i = \mathbf{0}$ for $i = 2, \cdots, n$.

Once more, linearity of the set of equations allows for solutions of the form $\vec{x}(t) = \vec{e}e^{\lambda t}$. As before, each subsequent derivative multiplies $\vec{x}(t)$ by a factor λ, such that

$$\frac{d^m \vec{x}}{dt^m} = \lambda^m \vec{x}(t) \,. \tag{2.3.2}$$

Substituting this result into Eq.(2.3.1) gives

$$\left[\mathbf{a}_m \lambda^m + \mathbf{a}_{m-1}\lambda^{m-1} + \cdots + \mathbf{a}_1 \lambda + \mathbf{a}_0\right] \vec{x}(t) \equiv \mathbf{D}(\lambda)\vec{x}(t) = 0 \,, \tag{2.3.3}$$

where $\mathbf{D}(\lambda)$ is an $n \times n$ matrix.

Equation (2.3.3) should be treated as follows:

- For each value of λ, the matrix $\mathbf{D}(\lambda)$ allows for n eigenvectors, such that

  $$\mathbf{D}(\lambda)\vec{E}^{\,\alpha}(\Lambda_\alpha) = \Lambda_\alpha(\lambda)\vec{E}^{\,\alpha}(\Lambda_\alpha) \quad \text{for} \quad \alpha = 1, 2, \cdots, n \,. \tag{2.3.4}$$

  (The direction of the eigenvector depends implicitly on λ through the explicit dependence of its eigenvalue.)

- The eigenvectors and eigenvalues of $\mathbf{D}$ vary with λ. For each α find solutions for λ to $\Lambda_\alpha(\lambda) = 0$. Since $\mathbf{D}$ is an mth order function of λ, there will be m such solutions for each α, i.e. a total of mn exponential rates, $\lambda_{\alpha,\ell}$ for $\alpha = 1, \cdots, n$ and $\ell = 1, \cdots, m$. These mn solutions are obtained by setting the determinant of $\mathbf{D}(\lambda)$ to zero.

- The appropriate eigenvectors for Eq. (2.3.3) are $\vec{e}^{\alpha} = \vec{E}^{\,\alpha}(0)$ evaluated at the m values of λ_α that satisfy $\Lambda_\alpha(\lambda_{\alpha,\ell}) = 0$.

- The general solution to Eq. (2.3.1) is then obtained as

  $$\vec{x}(t) = \sum_{\alpha=1}^{n} \left[\sum_{\ell=1}^{m} c_{\alpha,\ell} e^{\lambda_{\alpha,\ell} t}\right] \vec{e}^{\alpha} \,. \tag{2.3.5}$$