2.3.2 Normal modes

An important case of ODEs as in Eq. (2.3.1) is provided by generalization of the damped harmonic evolution in Eq. (1.4.7). For a collection of particles, whose deviations around a stable equilibrium point are indicated by x_i. The restoring forces in the ith direction for small amplitudes can be written as $F_i = -K_{ij}x_j$, where $K_{ij} = -\partial_i V$ is a symmetric matrix ($\mathbf{a}_0$ in Eq. (2.3.1)). The restoring force is balanced by mass times acceleration (appearing as $\mathbf{a}_2$ in Eq. (2.3.1)), and potentially frictional forces proportional to velocity ($\mathbf{a}_1$ in Eq. (2.3.1)), both with non-zero elements only along the diagonal.

For purposes of illustration, we further simplify the problem, setting all friction coefficients to zero $\mathbf{a}_1 = \mathbf{0}$, and all masses equal to unity $\mathbf{a}_2 = \mathbf{1}$, arriving to

$$\ddot{x}_i = -M_{ij}x_j\,, \tag{2.3.6}$$

with $M_{ij} = K_{ij}/m$.[7] For displacements around a stable equilibrium point, the matrix $\mathbf{K}$ is positive definite, and all eigenvalues $\{\lambda_\alpha\}$ of $\mathbf{M}$ are negative, An exponential decay for the first order (gradient descent) ODE translates to oscillations for the second order ODE of Eq. (2.3.6), at frequencies $-\omega_\alpha^2 = \lambda_\alpha$. In this context, the eigendirections are referred to as *normal modes* of the system, with $\{\omega_\alpha\}$ as the corresponding frequencies.

To solve for the solution to Eq. (2.3.6), we can follow the steps that lead to Eq. (2.2.12) for the first order ODEs. The important distinction is that for each normal mode, there are two frequencies $\pm\sqrt{-\lambda_\alpha}$, and time evolution along the corresponding eigendirection is oscillatory rather than exponential decay, i.e.

$$A_\alpha(t) = A_\alpha \cos(\omega_\alpha t + \phi_\alpha)\,. \tag{2.3.7}$$

Thus two parameters, e.g. the amplitude A_α and phase ϕ_α are needed to describe the contribution of the normal mode. The $2n$ parameters needed to characterize the full solution can for example be the initial displacements $x_i(0)$ and initial velocities $\dot{x}_i(0)$. Analogously to Eq. (2.2.12) a time evolution operator can be constructed to express the final positions and velocities in terms of the initial conditions, but this is beyond the scope of our interest.

[7] If the masses are not equal, we still arrive to Eq. (2.3.6), with $M_{ij} = K_{ij}/\sqrt{m_i m_j}$, after rescaling of $x_i \to x_i/\sqrt{m_i}$.