2.3.3 Normal modes of masses connected by springs

Symmetry plays a crucial role in constraining the form of eigenvectors of a matrix, and when symmetries are present they can greatly simplify the search for normal modes. We shall demonstrate the use of symmetries by considering normal modes of N identical blocks. The first block is connected by a spring on one side to a rigid wall, and by another spring to the second block on the other side. Each subsequent block is also connected on each side to the previous and next block, i.e. block m is connected to blocks $(m-1)$ and $(m+1)$. The last block is again connected to a rigid support. All the springs are assumed to be identical, with spring constant K, so that the overall potential energy is

$$V(x_1, \cdots, x_N) = V_0 + \frac{K}{2}\left[x_1^2 + (x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2 + x_N^2\right] . \qquad (2.3.8)$$

To write down the equations in compact form, let us introduce $x_0 = x_{N+1} = 0$, i.e. two immobile particles which will represent the two walls. We can then write

$$m\ddot{x}_i = F_i = -\frac{\partial V}{\partial x_i} = -K\left(2x_i - x_{i+1} - x_{i-1}\right) , \qquad \text{for } i = 2, \cdots, N-1 , \qquad (2.3.9)$$

with

$$m\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -K\left(2x_1 - x_2\right) , \quad \text{and} \quad m\ddot{x}_N = F_N = -\frac{\partial V}{\partial x_N} = -K\left(2x_N - x_{N-1}\right) . \qquad (2.3.10)$$

We again assume normal modes of the form

$$x_i(t) = A_i \cos(\omega t + \phi) , \qquad (2.3.11)$$

and substitute into the above equation to get

$$\omega^2 \vec{A} = \omega_0^2 \mathbf{T} \cdot \vec{A} , \qquad (2.3.12)$$

where $\omega_0 = \sqrt{K/m}$, and $\mathbf{T}$ is an $n \times n$ matrix whose elements are 2 along the diagonal ($T_{i,i} = 2$), -1 for each element that is next to a diagonal ($T_{i,i\pm 1} = -1$), and zero every where else. To find the normal mode frequencies, in units of ω_0, we just need to diagonalize the matrix $\mathbf{T}$. We shall do this first for 3 and 4 blocks, before going on to the general case.