Normal modes for 3 blocks

The matrix $\mathbf{T}$ in this case is

$$\mathbf{T}_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \tag{2.3.13}$$

Since the first and last particles are related by symmetry (reversing the order of particles) we expect normal modes in which these particles either move together, or in opposite directions. Indeed the easiest mode to guess is when these two particles move in opposite directions, and the central one is stationary, corresponding to

$$\vec{A}_2 = \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} . \tag{2.3.14}$$

It is easy to check that this is indeed an eigenvector, as

$$\mathbf{T}_3 \cdot \vec{A}_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} , \qquad \lambda_2 = 2 \, . \tag{2.3.15}$$

The corresponding frequency is $\sqrt{2}\omega_0$.

For the other eigenvectors, let us guess a form

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ +1 \end{pmatrix} , \tag{2.3.16}$$

and determine the parameter r. Since

$$\mathbf{T}_3 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ +1 \end{pmatrix} = \begin{pmatrix} 2-r \\ 2r-2 \\ 2-r \end{pmatrix} , \tag{2.3.17}$$

the form of the eigenvector is preserved if

$$\frac{2-r}{2r-2} = \frac{1}{r}, \qquad \Rightarrow r^2 - 2 = 0, \qquad \Rightarrow r = \pm\sqrt{2}. \tag{2.3.18}$$

We can indeed check that

$$\mathbf{T}_3 \cdot \vec{A}_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ \sqrt{2} \\ +1 \end{pmatrix} = (2-\sqrt{2}) \begin{pmatrix} +1 \\ \sqrt{2} \\ +1 \end{pmatrix} , \qquad \lambda_1 = 2-\sqrt{2}, \tag{2.3.19}$$

and

$$\mathbf{T}_3 \cdot \vec{A}_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ -\sqrt{2} \\ +1 \end{pmatrix} = (2+\sqrt{2}) \begin{pmatrix} +1 \\ -\sqrt{2} \\ +1 \end{pmatrix} , \qquad \lambda_3 = 2+\sqrt{2}. \tag{2.3.20}$$

The lowest frequency of the system, $\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$, is obtained when all three blocks move in the same direction, although the central one has a larger amplitude. The highest frequency, $\omega_1 = \omega_0\sqrt{2+\sqrt{2}}$, is obtained when the central block moves in the opposite direction.