Normal modes for 4 blocks

We have to diagonalize the 4×4 matrix

$$\mathbf{T}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{2.3.21}$$

In this case we can put the end-particles, and the central particles into two separate groups, each related by symmetry. Let's first guess an eigenvector of the form

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ r \\ +1 \end{pmatrix}, \tag{2.3.22}$$

which is symmetric under the reversal of label orders. From

$$\mathbf{T}_4 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ r \\ +1 \end{pmatrix} = \begin{pmatrix} 2-r \\ r-1 \\ r-1 \\ 2-r \end{pmatrix}, \tag{2.3.23}$$

we note that r has to be chosen such that

$$\frac{2-r}{r-1} = \frac{1}{r}, \qquad \Rightarrow r^2 - r - 1 = 0, \qquad \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}. \tag{2.3.24}$$

We can indeed then check that

$$\mathbf{T}_4 \cdot \vec{A}_1 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (1+\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ +1 \end{pmatrix} = \left(\frac{3-\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (1+\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ +1 \end{pmatrix}, \tag{2.3.25}$$

and

$$\mathbf{T}_4 \cdot \vec{A}_3 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (1-\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ +1 \end{pmatrix} = \left(\frac{3+\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (1-\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ +1 \end{pmatrix}. \tag{2.3.26}$$

(The eigenvectors are labelled in order of the magnitude of their eigenvalue.)

The other 2 eigenvalues are obtained by starting with

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ -r \\ -1 \end{pmatrix}, \tag{2.3.27}$$

which are antisymmetric under label reversal. Indeed this form is preserved, since

$$\mathbf{T}_4 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ -r \\ -1 \end{pmatrix} = \begin{pmatrix} 2-r \\ 3r-1 \\ -3r+1 \\ -2+r \end{pmatrix}, \tag{2.3.28}$$

if we choose r such that

$$\frac{2-r}{3r-1} = \frac{1}{r}, \qquad \Rightarrow r^2 + r - 1 = 0, \qquad \Rightarrow r = \frac{-1 \pm \sqrt{5}}{2}. \tag{2.3.29}$$

We can indeed then check that

$$\mathbf{T}_4 \cdot \vec{A}_2 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (-1+\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ -1 \end{pmatrix} = \left(\frac{5-\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (-1+\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ -1 \end{pmatrix}, \tag{2.3.30}$$

and

$$\mathbf{T}_4 \cdot \vec{A}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (-1-\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ -1 \end{pmatrix} = \left(\frac{5+\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (-1-\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ -1 \end{pmatrix}. \tag{2.3.31}$$

We have thus found all the normal modes in this case, and they are labelled in order of increasing frequencies. Again, the lowest frequency corresponds to particles moving most closely together, and the highest frequency to the motion in which the particles are most opposite each other.