2.3 Higher order coupled linear ODEs

2.3.1 General Form

The generalization of the mth order, linear, homogeneous ODE in Eq. (1.4.1) from a scalar $x(t)$ to an n-component vector $\vec{x}$ takes the form

$$\mathbf{a}_m \frac{d^m \vec{x}}{dt^m} + \mathbf{a}_{m-1} \frac{d^{m-1} \vec{x}}{dt^{m-1}} + \cdots + \mathbf{a}_1 \frac{d\vec{x}}{dt} + \mathbf{a}_0 \vec{x} = 0 \,, \tag{2.3.1}$$

where $\{\mathbf{a}_m, \cdots, \mathbf{a}_0\}$ are now $n \times n$ matrices. Note that the first order ODE set of last section, e.g. in Eq. (2.2.2), correspond to the choice of $\mathbf{a}_0 = \mathbf{M}$, $\mathbf{a}_1 = \mathbf{1}$, and $\mathbf{a}_i = \mathbf{0}$ for $i = 2, \cdots, n$.

Once more, linearity of the set of equations allows for solutions of the form $\vec{x}(t) = \vec{e} e^{\lambda t}$. As before, each subsequent derivative multiplies $\vec{x}(t)$ by a factor λ, such that

$$\frac{d^m \vec{x}}{dt^m} = \lambda^m \vec{x}(t) \,. \tag{2.3.2}$$

Substituting this result into Eq.(2.3.1) gives

$$\left[\mathbf{a}_m \lambda^m + \mathbf{a}_{m-1} \lambda^{m-1} + \cdots + \mathbf{a}_1 \lambda + \mathbf{a}_0\right] \vec{x}(t) \equiv \mathbf{D}(\lambda) \vec{x}(t) = 0 \,, \tag{2.3.3}$$

where $\mathbf{D}(\lambda)$ is an $n \times n$ matrix.

Equation (2.3.3) should be treated as follows:

- For each value of λ, the matrix $\mathbf{D}(\lambda)$ allows for n eigenvectors, such that

$$\mathbf{D}(\lambda) \vec{E}^{\alpha}(\Lambda_\alpha) = \Lambda_\alpha(\lambda) \vec{E}^{\alpha}(\Lambda_\alpha) \quad \text{for} \quad \alpha = 1, 2, \cdots, n \,. \tag{2.3.4}$$

  (The direction of the eigenvector depends implicitly on λ through the explicit dependence of its eigenvalue.)

- The eigenvectors and eigenvalues of $\mathbf{D}$ vary with λ. For each α find solutions for λ to $\Lambda_\alpha(\lambda) = 0$. Since $\mathbf{D}$ is an mth order function of λ, there will be m such solutions for each α, i.e. a total of mn exponential rates, $\lambda_{\alpha,\ell}$ for $\alpha = 1, \cdots, n$ and $\ell = 1, \cdots, m$. These mn solutions are obtained by setting the determinant of $\mathbf{D}(\lambda)$ to zero.

- The appropriate eigenvectors for Eq. (2.3.3) are $\vec{e}^{\alpha} = \vec{E}^{\alpha}(0)$ evaluated at the m values of λ_α that satisfy $\Lambda_\alpha(\lambda_{\alpha,\ell}) = 0$.

- The general solution to Eq. (2.3.1) is then obtained as

$$\vec{x}(t) = \sum_{\alpha=1}^{n} \left[\sum_{\ell=1}^{m} c_{\alpha,\ell} e^{\lambda_{\alpha,\ell} t} \right] \vec{e}^{\alpha} \,. \tag{2.3.5}$$

2.3.2 Normal modes

An important case of ODEs as in Eq. (2.3.1) is provided by generalization of the damped harmonic evolution in Eq. (1.4.7). For a collection of particles, whose deviations around a stable equilibrium point are indicated by x_i. The restoring forces in the ith direction for small amplitudes can be written as $F_i = -K_{ij}x_j$, where $K_{ij} = -\partial_i V$ is a symmetric matrix ($\mathbf{a}_0$ in Eq. (2.3.1)). The restoring force is balanced by mass times acceleration (appearing as $\mathbf{a}_2$ in Eq. (2.3.1)), and potentially frictional forces proportional to velocity ($\mathbf{a}_1$ in Eq. (2.3.1)), both with non-zero elements only along the diagonal.

For purposes of illustration, we further simplify the problem, setting all friction coefficients to zero $\mathbf{a}_1 = \mathbf{0}$, and all masses equal to unity $\mathbf{a}_2 = \mathbf{1}$, arriving to

$$\ddot{x}_i = -M_{ij}x_j \,, \tag{2.3.6}$$

with $M_{ij} = K_{ij}/m$.[7] For displacements around a stable equilibrium point, the matrix $\mathbf{K}$ is positive definite, and all eigenvalues $\{\lambda_\alpha\}$ of $\mathbf{M}$ are negative, An exponential decay for the first order (gradient descent) ODE translates to oscillations for the second order ODE of Eq. (2.3.6), at frequencies $-\omega_\alpha^2 = \lambda_\alpha$. In this context, the eigendirections are referred to as *normal modes* of the system, with $\{\omega_\alpha\}$ as the corresponding frequencies.

To solve for the solution to Eq. (2.3.6), we can follow the steps that lead to Eq. (2.2.12) for the first order ODEs. The important distinction is that for each normal mode, there are two frequencies $\pm\sqrt{-\lambda_\alpha}$, and time evolution along the corresponding eigendirection is oscillatory rather than exponential decay, i.e.

$$A_\alpha(t) = A_\alpha \cos(\omega_\alpha t + \phi_\alpha) \,. \tag{2.3.7}$$

Thus two parameters, e.g. the amplitude A_α and phase ϕ_α are needed to describe the contribution of the normal mode. The $2n$ parameters needed to characterize the full solution can for example be the initial displacements $x_i(0)$ and initial velocities $\dot{x}_i(0)$. Analogously to Eq. (2.2.12) a time evolution operator can be constructed to express the final positions and velocities in terms of the initial conditions, but this is beyond the scope of our interest.

2.3.3 Normal modes of masses connected by springs

Symmetry plays a crucial role in constraining the form of eigenvectors of a matrix, and when symmetries are present they can greatly simplify the search for normal modes. We shall demonstrate the use of symmetries by considering normal modes of N identical blocks. The first block is connected by a spring on one side to a rigid wall, and by another spring to the second block on the other side. Each subsequent block is also connected on each side to the previous and next block, i.e. block m is connected to blocks $(m-1)$ and $(m+1)$. The last block is again connected to a rigid support. All the springs are assumed to be identical, with

[7] If the masses are not equal, we still arrive to Eq. (2.3.6), with $M_{ij} = K_{ij}/\sqrt{m_i m_j}$, after rescaling of $x_i \to x_i/\sqrt{m_i}$.

spring constant K, so that the overall potential energy is

$$V(x_1, \cdots, x_N) = V_0 + \frac{K}{2}\left[x_1^2 + (x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2 + x_N^2\right]. \tag{2.3.8}$$

To write down the equations in compact form, let us introduce $x_0 = x_{N+1} = 0$, i.e. two immobile particles which will represent the two walls. We can then write

$$m\ddot{x}_i = F_i = -\frac{\partial V}{\partial x_i} = -K(2x_i - x_{i+1} - x_{i-1}), \qquad \text{for } i = 2, \cdots, N-1, \tag{2.3.9}$$

with

$$m\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -K(2x_1 - x_2), \quad \text{and} \quad m\ddot{x}_N = F_N = -\frac{\partial V}{\partial x_N} = -K(2x_N - x_{N-1}). \tag{2.3.10}$$

We again assume normal modes of the form

$$x_i(t) = A_i \cos(\omega t + \phi), \tag{2.3.11}$$

and substitute into the above equation to get

$$\omega^2 \vec{A} = \omega_0^2 \mathbf{T} \cdot \vec{A}, \tag{2.3.12}$$

where $\omega_0 = \sqrt{K/m}$, and $\mathbf{T}$ is an $n \times n$ matrix whose elements are 2 along the diagonal ($T_{i,i} = 2$), -1 for each element that is next to a diagonal ($T_{i,i\pm1} = -1$), and zero every where else. To find the normal mode frequencies, in units of ω_0, we just need to diagonalize the matrix $\mathbf{T}$. We shall do this first for 3 and 4 blocks, before going on to the general case.

Normal modes for 3 blocks

The matrix $\mathbf{T}$ in this case is

$$\mathbf{T}_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{2.3.13}$$

Since the first and last particles are related by symmetry (reversing the order of particles) we expect normal modes in which these particles either move together, or in opposite directions. Indeed the easiest mode to guess is when these two particles move in opposite directions, and the central one is stationary, corresponding to

$$\vec{A}_2 = \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}. \tag{2.3.14}$$

It is easy to check that this is indeed an eigenvector, as

$$\mathbf{T}_3 \cdot \vec{A}_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}, \qquad \lambda_2 = 2. \tag{2.3.15}$$

The corresponding frequency is $\sqrt{2}\omega_0$.

For the other eigenvectors, let us guess a form

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ +1 \end{pmatrix} , \tag{2.3.16}$$

and determine the parameter r. Since

$$\mathbf{T}_3 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ +1 \end{pmatrix} = \begin{pmatrix} 2-r \\ 2r-2 \\ 2-r \end{pmatrix} , \tag{2.3.17}$$

the form of the eigenvector is preserved if

$$\frac{2-r}{2r-2} = \frac{1}{r}, \qquad \Rightarrow r^2 - 2 = 0, \qquad \Rightarrow r = \pm\sqrt{2}. \tag{2.3.18}$$

We can indeed check that

$$\mathbf{T}_3 \cdot \vec{A}_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ \sqrt{2} \\ +1 \end{pmatrix} = (2-\sqrt{2}) \begin{pmatrix} +1 \\ \sqrt{2} \\ +1 \end{pmatrix} , \qquad \lambda_1 = 2-\sqrt{2}, \tag{2.3.19}$$

and

$$\mathbf{T}_3 \cdot \vec{A}_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ -\sqrt{2} \\ +1 \end{pmatrix} = (2+\sqrt{2}) \begin{pmatrix} +1 \\ -\sqrt{2} \\ +1 \end{pmatrix} , \qquad \lambda_3 = 2+\sqrt{2}. \tag{2.3.20}$$

The lowest frequency of the system, $\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$, is obtained when all three blocks move in the same direction, although the central one has a larger amplitude. The highest frequency, $\omega_1 = \omega_0\sqrt{2+\sqrt{2}}$, is obtained when the central block moves in the opposite direction.

Normal modes for 4 blocks

We have to diagonalize the 4×4 matrix

$$\mathbf{T}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} . \tag{2.3.21}$$

In this case we can put the end-particles, and the central particles into two separate groups, each related by symmetry. Let's first guess an eigenvector of the form

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ r \\ +1 \end{pmatrix} , \tag{2.3.22}$$

which is symmetric under the reversal of label orders. From

$$\mathbf{T}_4 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ r \\ +1 \end{pmatrix} = \begin{pmatrix} 2-r \\ r-1 \\ r-1 \\ 2-r \end{pmatrix}, \tag{2.3.23}$$

we note that r has to be chosen such that

$$\frac{2-r}{r-1} = \frac{1}{r}, \qquad \Rightarrow r^2 - r - 1 = 0, \qquad \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}. \tag{2.3.24}$$

We can indeed then check that

$$\mathbf{T}_4 \cdot \vec{A}_1 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (1+\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ +1 \end{pmatrix} = \left(\frac{3-\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (1+\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ +1 \end{pmatrix}, \tag{2.3.25}$$

and

$$\mathbf{T}_4 \cdot \vec{A}_3 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (1-\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ +1 \end{pmatrix} = \left(\frac{3+\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (1-\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ +1 \end{pmatrix}. \tag{2.3.26}$$

(The eigenvectors are labelled in order of the magnitude of their eigenvalue.)

The other 2 eigenvalues are obtained by starting with

$$\vec{A} = \begin{pmatrix} +1 \\ r \\ -r \\ -1 \end{pmatrix}, \tag{2.3.27}$$

which are antisymmetric under label reversal. Indeed this form is preserved, since

$$\mathbf{T}_4 \cdot \vec{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ r \\ -r \\ -1 \end{pmatrix} = \begin{pmatrix} 2-r \\ 3r-1 \\ -3r+1 \\ -2+r \end{pmatrix}, \tag{2.3.28}$$

if we choose r such that

$$\frac{2-r}{3r-1} = \frac{1}{r}, \qquad \Rightarrow r^2 + r - 1 = 0, \qquad \Rightarrow r = \frac{-1 \pm \sqrt{5}}{2}. \tag{2.3.29}$$

We can indeed then check that

$$\mathbf{T}_4 \cdot \vec{A}_2 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (-1+\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ -1 \end{pmatrix} = \left(\frac{5-\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (-1+\sqrt{5})/2 \\ (1-\sqrt{5})/2 \\ -1 \end{pmatrix}, \quad (2.3.30)$$

and

$$\mathbf{T}_4 \cdot \vec{A}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +1 \\ (-1-\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ -1 \end{pmatrix} = \left(\frac{5+\sqrt{5}}{2}\right) \begin{pmatrix} +1 \\ (-1-\sqrt{5})/2 \\ (1+\sqrt{5})/2 \\ -1 \end{pmatrix}. \quad (2.3.31)$$

We have thus found all the normal modes in this case, and they are labelled in order of increasing frequencies. Again, the lowest frequency corresponds to particles moving most closely together, and the highest frequency to the motion in which the particles are most opposite each other.

Normal modes for N blocks

It appears a daunting task to find all the normal modes for the general case of n blocks. However, there is a simple formula that generates all normal modes and frequencies. For the αth normal mode, we will try eigenvectors of the form

$$\vec{A}_\alpha = \begin{pmatrix} \sin\left(\frac{\pi\alpha}{N+1}1\right) \\ \sin\left(\frac{\pi\alpha}{N+1}2\right) \\ \vdots \\ \sin\left(\frac{\pi\alpha}{N+1}(N-1)\right) \\ \sin\left(\frac{\pi\alpha}{N+1}N\right) \end{pmatrix}, \quad \text{for} \quad \alpha = 1, 2, \cdots, N. \quad (2.3.32)$$

When multiplied by $\mathbf{T}_n$, a typical element is

$$\begin{aligned} &2\sin\left(\frac{\pi\alpha}{N+1}k\right) - \sin\left(\frac{\pi\alpha}{N+1}(k+1)\right) - \sin\left(\frac{\pi\alpha}{N+1}(k-1)\right) = \\ &2\sin\left(\frac{\pi\alpha}{N+1}k\right)\left[1-\cos\left(\frac{\pi\alpha}{N+1}\right)\right], \end{aligned} \quad (2.3.33)$$

where the trigonometric identity in Eq. (1.3.11) was used to convert the sum of two sines to the product of sine and cosine appearing in the second line. From this we can identify the normal mode frequencies

$$\omega_\alpha = \omega_0\sqrt{2\left[1-\cos\left(\frac{\pi\alpha}{N+1}\right)\right]} = 2\omega_0 \sin\left(\frac{\pi\alpha}{2(N+1)}\right), \quad (2.3.34)$$

by taking advantage of another trigonometric identity, $1-\cos(2\theta) = 2\sin^2\theta$. The reasoning that allows us to identify the eigenvector in Eq. (2.3.32) is explained in the next section.

Recap

- Normal modes in a potential $V = K_{ij}x_i x_j/2$, are eigenvectors of the symmetric matrix with elements $M_{ij} = K_{ij}/\sqrt{m_i m_j}$.
- The normal mode frequencies are related to the eigenvalues of the matrix, and obtained from solutions to $\det(\mathbf{M} - \omega^2 \mathbf{1}) = 0$.
- For a frictionless chain of n blocks connected in series by springs, the equations of motion

$$\ddot{x}_i = \omega_0^2 \left(x_{i+1} + x_{i-1} - 2x_i\right), \qquad \text{for } i = 1, 2, \cdots, n\,, \tag{2.3.35}$$

allow for normal modes

$$x_i^{(\alpha)}(t) = a_\alpha \sin\left(\frac{\pi \alpha i}{N+1}\right) \cos\left(\omega_\alpha t + \phi_\alpha\right), \qquad \text{for } \alpha = 1, 2, \cdots, n\,, \tag{2.3.36}$$

with frequenceis

$$\omega_\alpha^2 = 2\omega_0^2 \left[1 - \cos\left(\frac{\pi\alpha}{N+1}\right)\right], \quad \Rightarrow \quad \omega_\alpha = 2\omega_0 \sin\left(\frac{\pi\alpha}{2(N+1)}\right). \tag{2.3.37}$$