2.4 Symmetries in matrices

2.4.1 Exchange symmetry

In general, finding eigenvalues of an $n \times n$ matrix requires solving an nth order algebraic equation. Yet we found the eigenvalues of the 3×3 matrix $\mathbf{T}_3$ in Eq. (2.3.17) and the 4×4 matrix $\mathbf{T}_4$ in Eq. (2.3.21) using only quadratic equations. This is because we took advantage of symmetries of the system, e.g. by guessing that in the normal modes the two corner particles must move together or opposite each other. We would like to present a more formal approach to using symmetries to help diagonalize (find eigenvectors and eigenvalues) of a matrix.

The symmetry used in finding normal modes of 3 blocks is related to the invariance of the equations under relabelling of coordinates 1 and 3. For example

$$V(x_1, x_2, x_3) = V_0 + \frac{K}{2}\left[x_1^2 + (x_2 - x_1)^2 + x_3^2\right] = V(x_3, x_2, x_1)\ . \tag{2.4.1}$$

We can construct a matrix E_{13} that exchanges these labels on any vector, changing (v_1, v_2, v_3) to (v_3, v_2, v_1); this matrix is

$$\mathbf{E}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ . \tag{2.4.2}$$

It is now possible to check that

$$\mathbf{E}_{13}\mathbf{T}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{T}_3\mathbf{E}_{13}, \tag{2.4.3}$$

i.e. the dynamic matrix $\mathbf{T}_3$ commutes with the symmetry matrix $\mathbf{E}_{13}$. It can be shown quite generally that if a quadratic form $x_i K_{ij} x_j$ is invariant under a permutation (relabelling) of the indices, the corresponding symmetry matrix $\mathbf{P}$ commutes with the matrix $\mathbf{K}$.

We can now take advantage of the following important result:

• *If two (or more) matrices commute with each other, they can be simultaneously diagonalized*, i.e. they share the same eigenvectors (with different eigenvalues).

To prove this result, let us assume that matrices $\mathbf{P}$ and $\mathbf{K}$ commute, i.e. $\mathbf{PK} = \mathbf{KP}$, and that vector $\vec{e}$ is a non-degenerate[8] eigenvector of $\mathbf{P}$ with eigenvalue λ_P. We then have

$$\mathbf{P} \cdot (\mathbf{K}\vec{e}) = \mathbf{K} \cdot (\mathbf{P}\vec{e}) = \lambda_P(\mathbf{K}\vec{e})\,, \tag{2.4.4}$$

i.e. the vector $(\mathbf{K}\vec{e})$ is also an eigenvector of $\mathbf{P}$ with eigenvalue λ_P. Since this eigenvalue is non-degenerate, $(\mathbf{K}\vec{e})$ must be proportional to $\vec{e}$, namely

$$\mathbf{K}\vec{e} = \lambda_K \vec{e}\,, \tag{2.4.5}$$

[8] An eigenvector is *non-degenerate* if no other eigenvectors share the same eigenvalue. The proof can be easily generalized to allow for degenerate eigenvectors.

indicating that $\vec{e}$ is also an eigenvector of $\mathbf{K}$ with some (to be determined) eigenvalue λ_K.

It is usually simpler to diagonalize the matrix characterizing a symmetry. For example since two exchanges lead back to the original labeling, the exchange matrix satisfies $\mathbf{E}_{13}\cdot\mathbf{E}_{13} = \mathbf{1}$. Acting on an eigenvector of $\mathbf{E}_{13}$ with eigenvalue λ, this identify implies $\lambda^2 = 1$, i.e. the eigenvalues of $\mathbf{E}_{13}$ are ± 1. Knowledge of the eigenvalues enables construction of the corresponding eigenvectors. Let us denote the components of the eigenvector corresponding to $\lambda = -1$ by (e_1, e_2, e_3); then

$$\mathbf{T}_3 \cdot \vec{e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_2 \\ e_1 \end{pmatrix} = - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} , \quad \Rightarrow \quad e_3 = -e_1 \quad \text{and} \quad e_2 = 0\,. \tag{2.4.6}$$

We thus recover the eigenvector $\mathbf{A}_2$ proposed in Eq. (2.3.14). A similar construction for the eigenvector corresponding to $\lambda_+ = 1$ leads to

$$\mathbf{T}_3 \cdot \vec{e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} , \quad \Rightarrow \quad e_3 = e_1 \quad \text{and arbitrary} \quad e_2\,, \tag{2.4.7}$$

i.e. the eigenvector proposed in Eq. (2.3.16). As an exercise you can repeat the above analysis for the case of 4 blocks.