2.4.2 Periodic chain of blocks

To gain insight onto how the eigenvectors $\mathbf{A}_k$ of Eq. (2.3.32) were arrived at, we first start with a different problem. Consider a chain of N blocks connected by identical springs, including a spring connecting the endpoints to form a ring. The potential energy analogous to Eq. (2.3.8) is now

$$V(x_1, \cdots, x_N) = V_0 + \frac{K}{2}\left[(x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2 + (x_1 - x_N)^2\right]. \tag{2.4.8}$$

The ordering of coordinates along the ring is arbitrary, and we could have started counting from any one of the blocks. The symmetry can be captured by the $N \times N$ matrix

$$\mathbf{S}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.4.9}$$

corresponding to increasing all indices up by 1 (modulus N, i.e. $x_{N+1} = x_1$). We could have also shifted the labels by 2, corresponding to $\mathbf{S}_2 = \mathbf{S}_1 \cdot \mathbf{S}_1 = \mathbf{S}_1^2$. The collection of shift matrices $\{\mathbf{S}_p = \mathbf{S}_1^p\}$ commute with each other and are simultaneously diagonalizable. The possible eigenvalues are easily obtained by noting that $\mathbf{S}_N = \mathbf{S}_1^N = \mathbf{1}$, returning the original ordering; consequently

$$\lambda^N = 1 \quad \Longrightarrow \quad \lambda_\alpha = \exp\left(\frac{2\pi i \alpha}{N}\right) \equiv \omega^\alpha \quad \text{for } \alpha = 1, \cdots, N. \tag{2.4.10}$$

The complex solutions, referred to as the Nth roots of unity correspond to points in the complex plane at angles separated by $2\pi/N$. The eigenvalues appear in complex conjugate pairs, with the exception of 1 (and -1 for even N).

With eigenvalues at hand, we can proceed to constructing the eigenvectors of $\mathbf{S}_1$. For the αth eigenvector, $\mathbf{S}_1 \cdot \vec{e}^{\,\alpha} = \omega^\alpha \vec{e}^{\,\alpha}$ implies

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} e_1^\alpha \\ e_2^\alpha \\ e_3^\alpha \\ \vdots \\ e_N^\alpha \end{pmatrix} = \begin{pmatrix} e_2^\alpha \\ e_3^\alpha \\ e_4^\alpha \\ \vdots \\ e_1^\alpha \end{pmatrix} = \omega^\alpha \begin{pmatrix} e_1^\alpha \\ e_2^\alpha \\ e_3^\alpha \\ \vdots \\ e_N^\alpha \end{pmatrix}, \quad \Longrightarrow \quad e_k^\alpha = \omega^\alpha e_{k-1}^\alpha, \tag{2.4.11}$$

for $k = 2, \cdots, N$, while $e_1^\alpha = \omega^\alpha e_N^\alpha$. Taking advantage of $\omega^\alpha \omega^{(N-1)\alpha} = \exp(2\pi i \alpha) = 1$, we conclude that $e_k^\alpha = \omega^{\alpha(k-1)} e_1^\alpha$. Starting with $e_1^\alpha = \omega^\alpha$, and equiring the normalization

$(\vec{e}^{\alpha})^* \cdot \vec{e}^{\beta} = \delta_{\alpha\beta}$, then leads to the orthonormal set of vectors

$$\vec{e}^{\alpha} = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^{\alpha} \\ \omega^{2\alpha} \\ \omega^{3\alpha} \\ \vdots \\ \omega^{N\alpha} \end{pmatrix}, \text{ for } \alpha = 1, 2 \cdots, N, \text{ with } \omega = \exp\left(\frac{2\pi i}{N}\right). \tag{2.4.12}$$

Armed with the knowledge of eigenvectors, we can now evaluate the normal mode frequencies of the periodic chain according to Eq. (2.3.12), using the matrix

$$\mathbf{T}'_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1- & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}. \tag{2.4.13}$$

A typical element arising from $\mathbf{T}'_N \vec{e}^{\alpha} = \lambda_{\alpha} \vec{e}^{\alpha}$ gives

$$-\omega^{\alpha(k-1)} + 2\omega^{\alpha k} - \omega^{\alpha(k+1)} = \lambda_{\alpha} \omega^{\alpha k}, \tag{2.4.14}$$

leading to

$$\lambda_{\alpha} = -\omega^{\alpha} - \omega^{-\alpha} + 2 = 2 - 2\cos\left(\frac{\pi\alpha}{N}\right) = 4\sin^2\left(\frac{\pi\alpha}{2N}\right), \tag{2.4.15}$$

again using the identity $1 - \cos(2\theta) = 2\sin^2\theta$.

Note that while the eigenvalues of the asymmetric matrix $\mathbf{S}_1$ are complex, the corresponding eigenvalues for $\mathbf{T}'_N$ are real as required by its symmetry. This is accompanied by a degeneracy, since pairs of complex eigenvectors, corresponding to $\vec{e}^{\alpha}$ and $(\vec{e}^{\alpha})^*$ (indexed by α and $N - \alpha$), result in the same eigenvalue in Eq. (2.4.15). Due to this degeneracy, for each such pair, we can replace occurrences of $\omega^{\alpha k}$ and $\omega^{-\alpha k}$ in components of the complex eigenvectors in Eq. (2.4.12) with $\sin(\pi\alpha k/N)$ or $\cos(\pi\alpha k/N)$ to construct real eigenvectors,

$$\vec{c}^{\alpha} = \sqrt{\frac{2}{N}} \begin{pmatrix} \cos\left(\frac{2\pi\alpha}{N}1\right) \\ \cos\left(\frac{2\pi\alpha}{N}2\right) \\ \cos\left(\frac{2\pi\alpha}{N}3\right) \\ \vdots \\ \cos\left(\frac{2\pi\alpha}{N}N\right) = 1 \end{pmatrix}, \text{ and } \vec{s}^{\alpha} = \sqrt{\frac{2}{N}} \begin{pmatrix} \sin\left(\frac{2\pi\alpha}{N}1\right) \\ \sin\left(\frac{2\pi\alpha}{N}2\right) \\ \sin\left(\frac{2\pi\alpha}{N}3\right) \\ \vdots \\ \sin\left(\frac{2\pi\alpha}{N}N\right) = 0 \end{pmatrix}. \tag{2.4.16}$$

(To properly normalize such eigenvectors $\sqrt{1/N}$ will need to be replaced with $\sqrt{2/N}$.) We have to be careful with allowed values of α to avoid over-counting: $\alpha = 0, 1, 2, \cdots$ up to the integer part of $N/2$, with the $\alpha = 0$ (and $\alpha = N/2$ if N is even) absent for the sine modes.