2.4 Symmetries in matrices

2.4.1 Exchange symmetry

In general, finding eigenvalues of an $n \times n$ matrix requires solving an nth order algebraic equation. Yet we found the eigenvalues of the 3×3 matrix $\mathbf{T}_3$ in Eq. (2.3.17) and the 4×4 matrix $\mathbf{T}_4$ in Eq. (2.3.21) using only quadratic equations. This is because we took advantage of symmetries of the system, e.g. by guessing that in the normal modes the two corner particles must move together or opposite each other. We would like to present a more formal approach to using symmetries to help diagonalize (find eigenvectors and eigenvalues) of a matrix.

The symmetry used in finding normal modes of 3 blocks is related to the invariance of the equations under relabelling of coordinates 1 and 3. For example

$$V(x_1, x_2, x_3) = V_0 + \frac{K}{2}\left[x_1^2 + (x_2 - x_1)^2 + x_3^2\right] = V(x_3, x_2, x_1) \ . \tag{2.4.1}$$

We can construct a matrix E_{13} that exchanges these labels on any vector, changing (v_1, v_2, v_3) to (v_3, v_2, v_1); this matrix is

$$\mathbf{E}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ . \tag{2.4.2}$$

It is now possible to check that

$$\mathbf{E}_{13}\mathbf{T}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{T}_3\mathbf{E}_{13}, \tag{2.4.3}$$

i.e. the dynamic matrix $\mathbf{T}_3$ commutes with the symmetry matrix $\mathbf{E}_{13}$. It can be shown quite generally that if a quadratic form $x_i K_{ij} x_j$ is invariant under a permutation (relabelling) of the indices, the corresponding symmetry matrix $\mathbf{P}$ commutes with the matrix $\mathbf{K}$.

We can now take advantage of the following important result:

• *If two (or more) matrices commute with each other, they can be simultaneously diagonalized*, i.e. they share the same eigenvectors (with different eigenvalues).

To prove this result, let us assume that matrices $\mathbf{P}$ and $\mathbf{K}$ commute, i.e. $\mathbf{PK} = \mathbf{KP}$, and that vector $\vec{e}$ is a non-degenerate[8] eigenvector of $\mathbf{P}$ with eigenvalue λ_P. We then have

$$\mathbf{P} \cdot (\mathbf{K}\vec{e}) = \mathbf{K} \cdot (\mathbf{P}\vec{e}) = \lambda_P(\mathbf{K}\vec{e}) \, , \tag{2.4.4}$$

i.e. the vector $(\mathbf{K}\vec{e})$ is also an eigenvector of $\mathbf{P}$ with eigenvalue λ_P. Since this eigenvalue is non-degenerate, $(\mathbf{K}\vec{e})$ must be proportional to $\vec{e}$, namely

$$\mathbf{K}\vec{e} = \lambda_K \vec{e} \, , \tag{2.4.5}$$

[8] An eigenvector is *non-degenerate* if no other eigenvectors share the same eigenvalue. The proof can be easily generalized to allow for degenerate eigenvectors.

indicating that $\vec{e}$ is also an eigenvector of $\mathbf{K}$ with some (to be determined) eigenvalue λ_K.

It is usually simpler to diagonalize the matrix characterizing a symmetry. For example since two exchanges lead back to the original labeling, the exchange matrix satisfies $\mathbf{E}_{13} \cdot \mathbf{E}_{13} = \mathbf{1}$. Acting on an eigenvector of $\mathbf{E}_{13}$ with eigenvalue λ, this identify implies $\lambda^2 = 1$, i.e. the eigenvalues of $\mathbf{E}_{13}$ are ± 1. Knowledge of the eigenvalues enables construction of the corresponding eigenvectors. Let us denote the components of the eigenvector corresponding to $\lambda = -1$ by (e_1, e_2, e_3); then

$$\mathbf{T}_3 \cdot \vec{e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_2 \\ e_1 \end{pmatrix} = - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \Rightarrow \quad e_3 = -e_1 \quad \text{and} \quad e_2 = 0\,. \tag{2.4.6}$$

We thus recover the eigenvector $\mathbf{A}_2$ proposed in Eq. (2.3.14). A similar construction for the eigenvector corresponding to $\lambda_+ = 1$ leads to

$$\mathbf{T}_3 \cdot \vec{e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_3 \\ e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \Rightarrow \quad e_3 = e_1 \quad \text{and arbitrary} \quad e_2\,, \tag{2.4.7}$$

i.e. the eigenvector proposed in Eq. (2.3.16). As an exercise you can repeat the above analysis for the case of 4 blocks.

2.4.2 Periodic chain of blocks

To gain insight onto how the eigenvectors $\mathbf{A}_k$ of Eq. (2.3.32) were arrived at, we first start with a different problem. Consider a chain of N blocks connected by identical springs, including a spring connecting the endpoints to form a ring. The potential energy analogous to Eq. (2.3.8) is now

$$V(x_1, \cdots, x_N) = V_0 + \frac{K}{2}\left[(x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2 + (x_1 - x_N)^2\right]. \tag{2.4.8}$$

The ordering of coordinates along the ring is arbitrary, and we could have started counting from any one of the blocks. The symmetry can be captured by the $N \times N$ matrix

$$\mathbf{S}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.4.9}$$

corresponding to increasing all indices up by 1 (modulus N, i.e. $x_{N+1} = x_1$). We could have also shifted the labels by 2, corresponding to $\mathbf{S}_2 = \mathbf{S}_1 \cdot \mathbf{S}_1 = \mathbf{S}_1^2$. The collection of shift matrices $\{\mathbf{S}_p = \mathbf{S}_1^p\}$ commute with each other and are simultaneously diagonalizable. The possible eigenvalues are easily obtained by noting that $\mathbf{S}_N = \mathbf{S}_1^N = \mathbf{1}$, returning the original ordering; consequently

$$\lambda^N = 1 \quad \Longrightarrow \quad \lambda_\alpha = \exp\left(\frac{2\pi i\alpha}{N}\right) \equiv \omega^\alpha \quad \text{for } \alpha = 1, \cdots, N\,. \tag{2.4.10}$$

The complex solutions, referred to as the Nth roots of unity correspond to points in the complex plane at angles separated by $2\pi/N$. The eigenvalues appear in complex conjugate pairs, with the exception of 1 (and -1 for even N).

With eigenvalues at hand, we can proceed to constructing the eigenvectors of $\mathbf{S}_1$. For the αth eigenvector, $\mathbf{S}_1 \cdot \vec{e}^{\,\alpha} = \omega^\alpha \vec{e}^{\,\alpha}$ implies

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} e_1^\alpha \\ e_2^\alpha \\ e_3^\alpha \\ \vdots \\ e_N^\alpha \end{pmatrix} = \begin{pmatrix} e_2^\alpha \\ e_3^\alpha \\ e_4^\alpha \\ \vdots \\ e_1^\alpha \end{pmatrix} = \omega^\alpha \begin{pmatrix} e_1^\alpha \\ e_2^\alpha \\ e_3^\alpha \\ \vdots \\ e_N^\alpha \end{pmatrix}, \quad \Longrightarrow \quad e_k^\alpha = \omega^\alpha e_{k-1}^\alpha\,, \tag{2.4.11}$$

for $k = 2, \cdots, N$, while $e_1^\alpha = \omega^\alpha e_N^\alpha$. Taking advantage of $\omega^\alpha \omega^{(N-1)\alpha} = \exp(2\pi i\alpha) = 1$, we conclude that $e_k^\alpha = \omega^{\alpha(k-1)} e_1^\alpha$. Starting with $e_1^\alpha = \omega^\alpha$, and equiring the normalization $(\vec{e}^{\,\alpha})^* \cdot \vec{e}^{\,\beta} = \delta_{\alpha\beta}$, then leads to the orthonormal set of vectors

$$\vec{e}^{\,\alpha} = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^\alpha \\ \omega^{2\alpha} \\ \omega^{3\alpha} \\ \vdots \\ \omega^{N\alpha} \end{pmatrix}, \text{ for } \alpha = 1, 2 \cdots, N\,, \text{ with } \omega = \exp\left(\frac{2\pi i}{N}\right). \tag{2.4.12}$$

Armed with the knowledge of eigenvectors, we can now evaluate the normal mode frequencies of the periodic chain according to Eq. (2.3.12), using the matrix

$$\mathbf{T}'_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1- & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}. \tag{2.4.13}$$

A typical element arising from $\mathbf{T}'_N \vec{e}^{\,\alpha} = \lambda_\alpha \vec{e}^{\,\alpha}$ gives

$$-\omega^{\alpha(k-1)} + 2\omega^{\alpha k} - \omega^{\alpha(k+1)} = \lambda_\alpha \omega^{\alpha k}\,, \tag{2.4.14}$$

leading to

$$\lambda_\alpha = -\omega^\alpha - \omega^{-\alpha} + 2 = 2 - 2\cos\left(\frac{\pi\alpha}{N}\right) = 4\sin^2\left(\frac{\pi\alpha}{2N}\right), \tag{2.4.15}$$

again using the identity $1 - \cos(2\theta) = 2\sin^2\theta$.

Note that while the eigenvalues of the asymmetric matrix $\mathbf{S}_1$ are complex, the corresponding eigenvalues for $\mathbf{T}'_N$ are real as required by its symmetry. This is accompanied by a degeneracy, since pairs of complex eigenvectors, corresponding to $\vec{e}^{\alpha}$ and $(\vec{e}^{\alpha})^*$ (indexed by α and $N-\alpha$), result in the same eigenvalue in Eq. (2.4.15). Due to this degeneracy, for each such pair, we can replace occurrences of $\omega^{\alpha k}$ and $\omega^{-\alpha k}$ in components of the complex eigenvectors in Eq. (2.4.12) with $\sin(\pi\alpha k/N)$ or $\cos(\pi\alpha k/N)$ to construct real eigenvectors,

$$\vec{c}^{\alpha} = \sqrt{\frac{2}{N}} \begin{pmatrix} \cos\left(\frac{2\pi\alpha}{N}1\right) \\ \cos\left(\frac{2\pi\alpha}{N}2\right) \\ \cos\left(\frac{2\pi\alpha}{N}3\right) \\ \vdots \\ \cos\left(\frac{2\pi\alpha}{N}N\right) = 1 \end{pmatrix}, \text{ and } \vec{s}^{\alpha} = \sqrt{\frac{2}{N}} \begin{pmatrix} \sin\left(\frac{2\pi\alpha}{N}1\right) \\ \sin\left(\frac{2\pi\alpha}{N}2\right) \\ \sin\left(\frac{2\pi\alpha}{N}3\right) \\ \vdots \\ \sin\left(\frac{2\pi\alpha}{N}N\right) = 0 \end{pmatrix}. \tag{2.4.16}$$

(To properly normalize such eigenvectors $\sqrt{1/N}$ will need to be replaced with $\sqrt{2/N}$.) We have to be careful with allowed values of α to avoid over-counting: $\alpha = 0, 1, 2, \cdots$ up to the integer part of $N/2$, with the $\alpha = 0$ (and $\alpha = N/2$ if N is even) absent for the sine modes.

2.4.3 Pinned chain of blocks

While the eigenvalues of the periodic chain in Eq. (2.4.15) are non-negative, there is one mode for $\alpha = N$ with $\lambda_N = 0$. This mode describes the free motion of the center of mass of the chain which does not experience any restoring force. Such a mode is absent for the chain with the 2 ends pinned to walls presented in Eq. (2.3.8). However, the eigenvectors in Eq. (2.3.32) and the eigenvalues in Eq. (2.3.34) are almost identical to those of the periodic chain. How is this possible?

Consider the sine eigenvector in Eq. (2.4.16) for a *longer* chain of $M = 2(N+1)$ blocks. Ignore the first N components of this vector that may or may not be non-zero. The $(N+1)$th component is $\sin\left(\frac{2\pi\alpha(N+1)}{2(N+1)}\right) = 0$ is zero for all integer α. There are then again N components, coinciding exactly with those in Eq. (2.3.32), since $\sin\left(\frac{2\pi\alpha(N+1+k)}{2(N+1)}\right) = \sin\left(\frac{2\pi\alpha k}{2(N+1)}\right)$. that may or may not be non-zero, followed by the last element of the sine eigenvector which is always zero. The mid-point and final zero components of this eigenvector act precisely as the stationary walls to which the end blocks are connected. Thus normal modes of the pinned chain are found embedded in normal modes of a longer periodic chain!

Recap

- If two matrices commute they share the same eigenvectors, but with possibly different eigenvalues.
- Some symmetries (e.g. under exchange, shift, or permutation) can be described by matrices.
- It is typically easier to diagonalize matrices corresponding to symmetries; their eigenvectors can then be used to construct eigenvectors for problems sharing those symmetries.
- Translation (shift) symmetries are diagonalized by sine and cosine modes, presaging Fourier decomposition.